MATH2048 Honours Linear Algebra II

Solution to Midterm Examination 1

1. Let $W_1 = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 : a_1 + a_2 - a_4 = 0,\ a_2 + a_3 = 0\}$ and
$W_2 = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 : a_1 + a_2 + 2a_3 + a_4 = 0,\ a_2 - a_4 = 0\}$.

   (a) Find a basis β_1 for W_1 and a basis β_2 for W_2.

   (b) Compute $\dim(W_1 + W_2)$ and use it to determine whether or not $\mathbb{R}^4 = W_1 \oplus W_2$.

Solution:

(a) For $(a_1, a_2, a_3, a_4) \in W_1$, we have $\begin{cases} a_3 = -a_2 \\ a_4 = a_1 + a_2 \end{cases}$.

Hence, we have

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ -a_2 \\ a_1 + a_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \text{ where } a_1, a_2 \in \mathbb{R}.$$

Thus, we have $\beta_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

For $(a_1, a_2, a_3, a_4) \in W_2$, we have $\begin{cases} a_1 = -2a_3 - 2a_4 \\ a_2 = a_4 \end{cases}$

Hence, we have

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} -2a_3 - 2a_4 \\ a_4 \\ a_3 \\ a_4 \end{bmatrix} = a_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } a_3, a_4 \in \mathbb{R}.$$

Thus, we have $\beta_2 = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(b) Using (a), considering the (4×4)-matrix which consisting all column vectors of β_1 and β_2:

$$\begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1+R_4} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[-R_2+R_4]{R_2+R_3} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \xrightarrow[-2R_3+R_4]{2R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the rank of the matrix is 3, hence we have $\dim(W_1 + W_2) = 3$.
Further notice that $\dim(\mathbb{R}^4) = 4 \neq \dim(W_1 + W_2) = 3$.
Thus, $\mathbb{R}^4 \neq W_1 \oplus W_2$.

2. Let $p_0(x) = x + 1$. Consider the following mapping

$$T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$$
$$p(x) \mapsto \begin{pmatrix} p(0) & p'(1) \\ (p_0 \cdot p)'(0) & \int_0^1 p(t)dt \end{pmatrix}$$

Let $\beta = \{1, x, x^2\}$ and $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be bases for $P_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ respectively.

(a) Show that T is a linear transformation.

(b) Compute $[T]_\beta^\gamma$. Please show your steps.

(c) Use the rank-nullity theorem to determine whether T is one-to-one. Please explain your answer with details.

Solution:

(a) Take $f, g \in P_2(\mathbb{R})$ and $\alpha \in \mathbb{R}$, then we have

$$\begin{aligned} T(\alpha f + g) &= \begin{pmatrix} (\alpha f + g)(0) & (\alpha f + g)'(1) \\ (p_0 \cdot (\alpha f + g))'(0) & \int_0^1 (\alpha f(t) + g(t))dt \end{pmatrix} \\ &= \begin{pmatrix} \alpha f(0) + g(0) & \alpha f'(1) + g'(1) \\ \alpha (p_0 \cdot f)'(0) + (p_0 \cdot g)'(0) & \int_0^1 \alpha f(t)dt + \int_0^1 g(t)dt \end{pmatrix} \\ &= \begin{pmatrix} \alpha f(0) & \alpha f'(1) \\ \alpha (p_0 \cdot f)'(0) & \alpha \int_0^1 f(t)dt \end{pmatrix} + \begin{pmatrix} g(0) & g'(1) \\ (p_0 \cdot g)'(0) & \int_0^1 g(t)dt \end{pmatrix} \\ &= \alpha T(f) + T(g) \end{aligned}$$

Thus, T is a linear transformation.

(b) Note that

$$T(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \mathbf{3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{0} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{1} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$T(x) = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} = \mathbf{\frac{5}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{1} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{\frac{1}{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & \frac{1}{3} \end{pmatrix} = \mathbf{\frac{7}{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathbf{2} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{0} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{\frac{1}{3}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, we have

$$[T]_\beta^\gamma = \begin{bmatrix} 3 & \frac{5}{2} & \frac{7}{3} \\ 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

(c) Using (b), note that

$$[T]_\beta^\gamma \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & \frac{5}{7} & \frac{7}{3} \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \xrightarrow[-3R_1+R_3]{-R_1+R_4} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -\frac{16}{7} & \frac{7}{3} \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{bmatrix} \xrightarrow[\frac{16}{7}R_2+R_3]{\frac{1}{2}R_2+R_4,-R_2+R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & \frac{7}{3} \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$$\xrightarrow{\frac{3}{7}R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \xrightarrow[-\frac{4}{3}R_3+R_4]{2R_3+R_1,-2R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence by rank-nullity theorem, we have

$$\begin{aligned} \text{rank}\left([T]_\beta^\gamma\right) + \text{Nullity}\left([T]_\beta^\gamma\right) &= \dim\left(P_2(\mathbb{R})\right) = 3 \\ 3 + \text{Nullity}\left([T]_\beta^\gamma\right) &= 3 \\ \text{Nullity}\left([T]_\beta^\gamma\right) &= 0 \end{aligned}$$

Thus, we have $\mathcal{N}\left([T]_\beta^\gamma\right) = \{0\}$ and this shows that T is one-to-one.

3. Let

$$V = \left\{ \sum_{m=1}^{K} a_m \sin(mx) + \sum_{n=1}^{K} b_n \cos(nx) : a_m, b_n \in \mathbb{R} \text{ for } m, n = 1, \ldots, K \right\}$$

be a vector space over $\mathbb{R}$. The addition and scalar multiplication are defined as $(af+g)(x) = af(x) + g(x)$ for any $f, g \in V$ and $a \in \mathbb{R}$.
Given $\beta = \{\sin(mx), \cos(nx)\}_{m,n=1}^{K}$ is a basis for V. Let $T : V \to V$ be defined as $T(f) := -f'' + f$, where f'' refers to the second order derivatives of f.

(a) Show that T is a linear transformation.

(b) Show that T is an isomorphism.

Solution:

(a) Take $f, g \in V$ and $a \in \mathbb{R}$, then we have

$$\begin{aligned} T\left(af+g\right) &= -(af+g)'' + (af+g) \\ &= -af'' - g'' + af + g \\ &= a(-f'' + f) + (-g'' + g) \\ &= aT(f) + T(g) \end{aligned}$$

Thus, T is a linear transformation.

(b) Now, it remains to show T is one-to-one and onto.
For any $f \in \mathcal{N}(T) \subset V$ such that $T(f) = 0$, let

$$f(x) = \sum_{m=1}^{K} a_m \sin(mx) + \sum_{n=1}^{K} b_n \cos(nx),$$

where $a_m, b_n \in \mathbb{R}$ and $m, n = 1, \ldots, K$.
Then, we have

$$\begin{aligned} T(f) &= -f'' + f \\ &= -\left(\sum_{m=1}^{K} -a_m m^2 \sin(mx) + \sum_{n=1}^{K} -b_n n^2 \cos(nx) \right) + \left(\sum_{m=1}^{K} a_m \sin(mx) + \sum_{n=1}^{K} b_n \cos(nx) \right) \\ &= \sum_{m=1}^{K} (1+m^2) a_m \sin(mx) + \sum_{n=1}^{K} (1+n^2) b_n \cos(nx) \end{aligned}$$

and hence

$$\sum_{m=1}^{K} (1+m^2) a_m \sin(mx) + \sum_{n=1}^{K} (1+n^2) b_n \cos(nx) = 0$$

$$(1+m^2) a_m = (1+n^2) b_n = 0$$

because $\beta = \{\sin(mx), \cos(nx)\}_{m,n=1}^{K}$ is a basis for V.
Note that $m, n = 1, \ldots, K \implies 1 + m^2, 1 + n^2 \neq 0$, hence we have $a_m = b_n = 0$ for all $m, n = 1, \ldots, K$. This implies that $f = 0$ and $\mathcal{N}(T) = \{0\}$, thus T is one-to-one.

Moreover, from the above $T(f)$ for any $f \in V$, it is clearly that $\mathcal{R}(T) = \mathsf{span}(\beta)$.
Hence $\dim \mathcal{R}(T) = |\beta| = 2K = \dim V$ and hence T is onto.
Thus, T is isomorphism as T is linear, one-to-one and onto.

4. Let $V = C([0,1], \mathbb{R})$ be the vector space of real-valued continuous functions on $[0,1]$.

(a) Let $\Phi : V \to \mathbb{R}^k$ be a linear transformation. Define the induced linear transformation $\widetilde{\Phi} : V/\mathcal{N}(\Phi) \to \mathbb{R}^k$ by $\widetilde{\Phi}(v + \mathcal{N}(\Phi)) = \Phi(v)$. Show that $\widetilde{\Phi}$ is an isomorphism if and only if Φ is onto.

(b) Let W be a subspace of V defined as follows:

$$W = \left\{ f \in V : f(0) = f\left(\frac{1}{N}\right) = f\left(\frac{2}{N}\right) = \cdots = f\left(\frac{N-1}{N}\right) \right\}.$$

Construct an isomorphism between V/W and $\mathbb{R}^k$, where $k = \dim\left(V/W\right)$. Deduce the dimension of V/W.

Solution:

(a) ($\Rightarrow$) Suppose $\widetilde{\Phi}$ is an isomorphism, then for any $\mathbf{y} \in \mathbb{R}^k$, there exists $v + \mathcal{N}(\Phi) \in V/\mathcal{N}(\Phi)$ for some $v \in V$ such that $\widetilde{\Phi}\left(v + \mathcal{N}(\Phi)\right) = \mathbf{y} = \Phi(v)$. Thus, we have Φ is onto.

($\Leftarrow$) Suppose Φ is onto, then for any $\mathbf{y} \in \mathbb{R}^k$, there exist some $v \in V$ such that $\mathbf{y} = \Phi(v) = \widetilde{\Phi}(v + \mathcal{N}(\Phi))$, which is clear that $\widetilde{\Phi}$ is onto. Now, it remains to show that $\widetilde{\Phi}$ is one-to-one.

For any $u + \mathcal{N}(\Phi) \in \mathcal{N}(\widetilde{\Phi})$, we have $\widetilde{\Phi}\left(u + \mathcal{N}(\Phi)\right) = \mathbf{0} = \Phi(u)$ and this implies that $u \in \mathcal{N}(\Phi)$, so we have $u + \mathcal{N}(\Phi) = 0 + \mathcal{N}(\Phi)$ and hence $\widetilde{\Phi}$ is one-to-one. Thus, we have $\widetilde{\Phi}$ is isomorphism.

(b) Note that we want to construct an isomorphism $\widetilde{\Phi}$ between V/W and $\mathbb{R}^k$, by using the result in (a), if we have to consider there is a linear map $\Phi : V \to \mathbb{R}^k$ such that $\widetilde{\Phi} : V/\mathcal{N}(\Phi) \to \mathbb{R}^k$ and such Φ must be onto. That means, for any $f \in W$, we have to construct such onto Φ and satisfies $\Phi(f) = \mathbf{0}$ and $W = \mathcal{N}(\Phi)$.

Construct $\Phi : V \to \mathbb{R}^{N-1}$ by $f \mapsto \begin{bmatrix} f\left(\frac{1}{N}\right) - f(0) \\ \vdots \\ f\left(\frac{N-1}{N}\right) - f(0) \end{bmatrix}$

It is obviously that Φ is linear.

Then, we want to show $W = \mathcal{N}(\Phi)$ makes sense.

For any $f \in \mathcal{N}(\Phi)$, we have

$$\mathbf{0} = \Phi(f) = \begin{bmatrix} f\left(\frac{1}{N}\right) - f(0) \\ \vdots \\ f\left(\frac{N-1}{N}\right) - f(0) \end{bmatrix}$$

then this follows that

$$f(0) = f\left(\frac{1}{N}\right) = f\left(\frac{2}{N}\right) = \cdots = f\left(\frac{N-1}{N}\right)$$

Hence, this shows that $f \in W$ and $\mathcal{N}(\Phi) \subset W$.

On the other hand, for any $f \in W$, we have

$$f(0) = f\left(\frac{1}{N}\right) = f\left(\frac{2}{N}\right) = \cdots = f\left(\frac{N-1}{N}\right)$$

then we have

$$f\left(\frac{1}{N}\right) - f(0) = \cdots = f\left(\frac{N-1}{N}\right) - f(0) = 0$$

and hence

$$\Phi(f) = \begin{bmatrix} f\left(\frac{1}{N}\right) - f(0) \\ \vdots \\ f\left(\frac{N-1}{N}\right) - f(0) \end{bmatrix} = \mathbf{0}$$

This shows that $f \in \mathcal{N}(\Phi)$ and $W \subset \mathcal{N}(\Phi)$.
Therefore, we have $W = \mathcal{N}(\Phi)$.

Next, it remains to show Φ is onto.

For any $\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} \in \mathbb{R}^{N-1}$, there is a piecewise linear function $f \in V$ defined as follows:

$$\begin{cases} f(0) = 0 \\ f\left(\frac{k}{N}\right) = a_k + f(0) \quad k = 1, \ldots, N-1 \end{cases}$$

such that $\Phi(f) = \mathbf{a}$.
Thus, Φ is onto.

Finally, by using (a), the induced linear transformation $\widetilde{\Phi} : V/\mathcal{N}(\Phi) \to \mathbb{R}^{N-1}$ is defined by $\widetilde{\Phi}(v + \mathcal{N}(\Phi)) = \Phi(v)$,
that is $\widetilde{\Phi} : V/W \to \mathbb{R}^{N-1}$ defined as $v + W \mapsto \Phi(v)$ is isomorphism follows from the result of (a).
Thus, it is clear that $\dim\left(V/W\right) = N - 1$.

5. Let V be an infinite dimensional vector space over F. Suppose W is a proper subspace of V (that is, $W \subsetneq V$). Consider the family of subspaces:

$$\mathscr{F} := \{A \subset V : A \text{ is a subspace and } A \cap W = \{\mathbf{0}\}\}.$$

(a) Using Zorn's lemma, prove that $\mathscr{F}$ contains a maximal element $\widetilde{W}$.

(b) Prove that $V = W \oplus \widetilde{W}$.

Solution:

(a) **First of all, the elements in $\mathscr{F}$ are partially ordered with respect to inclusion.**
For any chain $\mathscr{C}$ in $\mathscr{F}$: $A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$, define

$$\widetilde{A} = \bigcup_{k=1}^{\infty} A_k$$

Then, we have to show $\widetilde{A} \in \mathscr{F}$.

- Note that $\widetilde{A} \cap W = \left(\bigcup_{k=1}^{\infty} A_k \right) \cap W = \bigcup_{k=1}^{\infty} (A_k \cap W) = \bigcup_{k=1}^{\infty} \{\mathbf{0}\} = \{\mathbf{0}\}$
- Also, $\widetilde{A}$ is a subspace.

Since $\mathbf{0} \in A_0 \in \widetilde{A}$.
For any $\mathbf{x}, \mathbf{y} \in \widetilde{A}$, there exist $m, n \in \mathbb{Z}^+$ such that $\mathbf{x} \in A_m$ and $\mathbf{y} \in A_n$.
It implies that $\mathbf{x}, \mathbf{y} \in A_{\max\{m,n\}}$.
Hence $\alpha \mathbf{x} + \mathbf{y} \in A_{\max\{m,n\}} \subset \widetilde{A}$, for any $\alpha \in F$ and $\mathbf{x}, \mathbf{y} \in \widetilde{A}$.
Last, applying Zorn's lemma.
Since $\widetilde{A}$ is a member of $\mathscr{F}$ that contains each member of $\mathscr{C}$.
By Zorn's lemma, $\mathscr{F}$ contains a maximal element $\widetilde{W}$.

(b) Using the result of (a), since $\widetilde{W} \in \mathscr{F}$ and hence $\widetilde{W} \cap W = \{\mathbf{0}\}$.
Now, it remains to show that $V = W + \widetilde{W}$.
Since $W, \widetilde{W} \subset V$, obviously $W + \widetilde{W} \subset V$.
Suppose that $V \subsetneq W + \widetilde{W}$, then there exist some $\mathbf{x} \in V \setminus \left(W + \widetilde{W} \right)$ and $\mathbf{x} \neq \mathbf{0}$.
Now, it is sufficient to show $\left(\widetilde{W} + \mathsf{span}\{\mathbf{x}\} \right) \cap W = \{\mathbf{0}\}$.
For any $\mathbf{y} \in \left(\widetilde{W} + \mathsf{span}\{\mathbf{x}\} \right) \cap W$, there exist $\tilde{\mathbf{w}} \in \widetilde{W}$ and $a \in F$ such that

$$\mathbf{y} = \tilde{\mathbf{w}} + a\mathbf{x}$$

Since $\mathbf{y} \in W$, we have $a\mathbf{x} = \mathbf{y} - \tilde{\mathbf{w}} \in W + \widetilde{W}$.
However, $\mathbf{x} \notin W + \widetilde{W}$ and $\mathbf{x} \neq \mathbf{0}$, we have $a = 0$.
Hence, we have

$$\tilde{\mathbf{w}} = \tilde{\mathbf{w}} + 0\mathbf{x} = \mathbf{y} \in W$$

It implies that $\tilde{\mathbf{w}} \in W \cap \widetilde{W} = \{\mathbf{0}\}$ and $\tilde{\mathbf{w}} = \mathbf{0}$ and hence $\mathbf{y} = \tilde{\mathbf{w}} + a\mathbf{x} = \mathbf{0}$.
Therefore, we have $\left(\widetilde{W} + \mathsf{span}\{\mathbf{x}\} \right) \cap W = \{\mathbf{0}\}$ and then $\widetilde{W} + \mathsf{span}\{\mathbf{x}\} \in \mathscr{F}$.
This contradicts to the maximality of existence of $\widetilde{W}$ in $\mathscr{F}$. So, the assumption that $V \subsetneq W + \widetilde{W}$ is false and therefore $V = W + \widetilde{W}$.
Thus, by definition and this shows that $V = W \oplus \widetilde{W}$ and completes the proof.

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