Tutorial 5. Eigenvalues, Eigenspaces, Diagonalizability, Invariant Subspaces and Cayley-Hamilton Theorem.

We will focus on some insightful problems in this tutorial.

Q1. If a linear map $T: V \to V$ is nilpotent, i.e., $T^n = O$ (zero linear map) for some $n \in \mathbb{N}$, then all eigenvalues of T are 0.

pf. Suppose λ is its eigenvalue, with eigenvector $v \neq 0$. then

$$0 = T^n v = \lambda^n v$$

This implies $\lambda^n = 0$. Hence $\lambda = 0$. $\square$.

Q2 (i) Find two 2x2 matrices which have the same characteristic polynomial but not similar.

(ii) How about 4x4 matrices? $(\mathbb{F} = \mathbb{R})$

Ans: (i). $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\chi_{A_1} = \chi_{A_2} = (1-x)^2$. They are definitely not similar as $PIP^{-1} = I$. $\forall P \in M_{2\times 2}(\mathbb{R})$.

(ii). $A_1 = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ $A_2 = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$.

$\chi_{A_1} = \chi_{A_2} = (1-x)^4$.

But A_1 and A_2 are not similar as the geometric multiplicities of eigenvalue 1 are different.

$\dim\ker(A_1 - I) = 3$. $\dim\ker(A_2 - I) = 2$.

Def 5.1. The geometric multiplicity of an eigenvalue λ of a linear map $T: V \to V$ is

$$\dim\ker(T - \lambda\, Id).$$

The dimension of eigenspace V_λ.

Prop. 5.2. The geometric multiplicity is an invariant for similar matrices, i.e., for invertible P,

$$\dim\ker(A - \lambda I) = \dim\ker(PAP^{-1} - \lambda I).$$

pf. $v \in \ker(A - \lambda I) \Leftrightarrow Av = \lambda v \Leftrightarrow PAP^{-1}(Pv) = PAv = \lambda Pv \Leftrightarrow Pv \in \ker(PAP^{-1} - \lambda I)$.

As P is invertible, dim is preserved. $\square$.

[Algebraic multiplicity is the exponent m_i of λ_i in $\chi_T(x) = (x - \lambda_i)^{m_i}(\cdots)$, where χ_T is the characteristic polynomial.]

Q3. Determine the formula for Fibonacci number x_n by $x_{n+2} = x_{n+1} + x_n$, $x_0 = 0$, $x_1 = 1$, i.e., Find x_n in terms of n.

Ans. $X_{n+3} = X_{n+2} + X_{n+1} = 2X_{n+1} + X_n$

Written in matrix,
$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} X_n \\ X_{n+1} \end{pmatrix} = \begin{pmatrix} X_{n+2} \\ X_{n+3} \end{pmatrix}.$$

let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. For $n \geq 0$, we have
$$A^n \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} X_{2n} \\ X_{2n+1} \end{pmatrix}.$$

To compute A^n, we factorize A. The characteristic polynomial
$$\chi_A(x) = x^2 - 3x + 1$$
so A has eigenvalues
$$\lambda_1 = \frac{3+\sqrt{5}}{2} = \varphi^2 \quad \lambda_2 = \frac{3-\sqrt{5}}{2} = \varphi^{-2}$$
where $\varphi = \frac{1+\sqrt{5}}{2}$ is the Golden ratio.

and eigenvectors
$$v_1 = \begin{pmatrix} 1 \\ \varphi \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -\varphi^{-1} \end{pmatrix}.$$

So $\exists P$ invertible s.t. $P^{-1}AP = \begin{pmatrix} \varphi^2 & \\ & \varphi^{-2} \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 \\ \varphi & -\varphi^{-1} \end{pmatrix}$, $P^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi^{-1} & 1 \\ \varphi & -1 \end{pmatrix}$.

Therefore
$$A^n = P(P^{-1}AP)^n P^{-1} = P\begin{pmatrix} \varphi^{2n} & \\ & \varphi^{-2n} \end{pmatrix} P^{-1} = \begin{pmatrix} \varphi^{2n-1} + \varphi^{-2n+1} & \varphi^{2n} - \varphi^{-2n} \\ \varphi^{2n} - \varphi^{-2n} & \varphi^{2n+1} + \varphi^{-2n-1} \end{pmatrix}.$$

So
$$\begin{pmatrix} X_{2n} \\ X_{2n+1} \end{pmatrix} = A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{2n} - \varphi^{-2n} \\ \varphi^{2n+1} + \varphi^{-2n-1} \end{pmatrix}$$

$$X_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Remark. One may also do $\begin{pmatrix} X_{n+1} \\ X_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} X_n \\ X_{n+1} \end{pmatrix}$ significantly simplifying the process.

Q4. Condition 1: $\mathbb{F}=\mathbb{C}$ or algebraically closed field. ← every non-constant polynomial has a root in the field.

condition 2: $\dim V<\infty$. $V\neq 0$.

(i) If both conditions satisfied, any linear map $T: V\to V$ has a non-zero eigenvector.

(ii) Give counterexamples when either condition is omitted.

pf. (i) Let $n=\dim V<\infty$. Consider the set of $n+1$ vectors.

$$\{v, Tv, \cdots, T^n v\}$$

This must be linearly dependent as $n+1>\dim V$, so $\exists\, a_i\in\mathbb{C}$ s.t.

$$\sum_{i=0}^{n} a_i T^i v = 0.$$

Consider $f(x)=\sum_{i=0}^{n} a_i x^i$. By fundamental theorem of algebra,

$$f(x)=a_0(x-x_1)(x-x_2)\cdots(x-x_n) \quad \text{where } x_1,\cdots,x_n \text{ are roots of } f(x).$$

As $f(T)=0$ by above,

$$a_0(T-x_1 Id)(T-x_2 Id)\cdots(T-x_n Id)=0.$$

This means at least one of $T-x_i Id$ is not invertible (by taking determinant for example).

Therefore we have a corresponding eigenvector for this eigenvalue x_i.

(ii) If $\mathbb{F}=\mathbb{R}$. $T=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ for $\theta\notin 2\pi\mathbb{Z}$ has no eigenvectors, as it rotates all vectors in $\mathbb{R}^2$.

If $\mathbb{F}=\mathbb{C}$ and V is infinite dimensional, take

$$T(x_1,x_2,x_3,\cdots)=(0,x_1,x_2,x_3,\cdots) \quad \text{right-shift.}$$

Then we claim T has no eigenvectors.

Suppose $\underset{\substack{\| \\ (0,v_1,v_2,\cdots)}}{T(v_1,v_2,\cdots)} = \underset{\substack{\| \\ (\lambda v_1,\lambda v_2,\cdots)}}{\lambda(v_1,v_2,\cdots)}$ $\lambda\in\mathbb{C}$. $v=(v_1,v_2,\cdots)$ is an eigenvector.

Then $\lambda v_1=0$, $v_1=\lambda v_2$, $v_2=\lambda v_3$ etc.

If $\lambda\neq 0$, then $v=0$. $\Rightarrow\Leftarrow$.

If $\lambda=0$ then $v_1=\lambda v_2=0$, $v_2=\lambda v_3=0$ etc, $v=0$ $\Rightarrow\Leftarrow$.

Diagonalizability.

Def 5.3. Let V be a finite dimensional vector space. $T: V \to V$ a linear map.

(i) T is <u>diagonalizable</u> iff $\exists$ a basis β of V s.t. $[T]_\beta$ is diagonal.

(ii) β is called <u>eigenbasis</u> of T consisting of eigenvectors of T.

(iii). $E_{\lambda_i} = \ker(T - \lambda_i Id)$ is called <u>eigenspaces</u> of T with respect to eigenvalue λ_i.

Eg. $T: \mathbb{R}^5 \to \mathbb{R}^5$, $(x_1, x_2, x_3, x_4, x_5) \longmapsto (2x_1, 2x_2, 2x_3, 3x_4, 3x_5)$.

Then just choose standard basis $\beta = \{e_1, e_2, e_3, e_4, e_5\}$ of $\mathbb{R}^5$.

$$[T]_\beta = \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix} \qquad \text{so } \lambda_1 = 2, \ \lambda_2 = 3.$$

$$\begin{aligned} V &= E_{\lambda_1} \oplus E_{\lambda_2} \\ &= \operatorname{span}\{e_1, e_2, e_3\} \oplus \operatorname{span}\{e_4, e_5\} \end{aligned} \qquad [T]_\beta = \left(\begin{array}{c|c} [T|_{E_{\lambda_1}}]_\beta & \\ \hline & T|_{E_{\lambda_2}} \end{array}\right) \text{ is block diagonal.}$$

Prop. 5.4 Let $T: V \to V$ and $\dim V < \infty$. Suppose $\lambda_1, \cdots, \lambda_m$ are <u>distinct</u> eigenvalues of T

(i) T with eigenspaces E_{λ_i}: λ_i

(i) Each E_{λ_i} is T-invariant.

(ii) T is diagonalizable iff $V = \bigoplus_{i=1}^{m} E_{\lambda_i}$. ← span of corresponding eigenbasis of λ_i.

(iii). If $V = \bigoplus_{i=1}^{m} V_i$ where V_i are all T-invariant, then

T is diagonalizable $\Leftrightarrow$ $T|_{V_i}$ are diagonalizable for all i.

Remark. (iii) is to look at $[T]_\beta$ as blocks $[T|_{V_i}]_\beta$. It is obvious to see it in matrices.

Def. 5.5. (simultaneously diagonalizability).

Let V be a finite dimensional vector space and $T, S: V \to V$ two linear maps.

We say T, S are <u>simultaneously diagonalizable</u> if $\exists$ a basis β for V s.t. $[T]_\beta$ and $[S]_\beta$ are diagonal matrices.

same basis

[We say A, B are " " " " if $\exists P$ invertible s.t. $P^{-1}AP$ and $P^{-1}BP$ are diagonal]

Q5. In the setting above,

(i). If T, S are simultaneously diagonalizable, then $TS = ST$.

(ii) If T, S are diagonalizable and $TS = ST$, then T, S are simultaneously diagonalizable.

Pf. (i) As $[T]_\beta$, $[S]_\beta$ are diagonal matrices, $[T\circ S]_\beta = [T]_\beta [S]_\beta = [S]_\beta [T]_\beta = [S\circ T]_\beta$. so $TS = ST$.

(ii). By change of basis formula, T, S are simultaneously diagonalizable if we can find an eigenbasis β common to T and S. (P is the matrix consists of eigenbasis β)

Let $V = \bigoplus_{i=1}^m E_{\lambda_i}$ where $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T, following from prop 5.4 (ii) and T being diagonalizable. Now we claim $S(E_{\lambda_i}) \subseteq E_{\lambda_i}$. As for every $v \in E_{\lambda_i}$,

$$TSv \overset{\text{commutative}}{=} STv = S\lambda_i v = \lambda_i (Sv).$$

so $Sv \in \ker(T - \lambda_i I) = E_{\lambda_i}$. Now by Prop 5.4 (iii), $S|_{E_{\lambda_i}}$ is diagonalizable.

Take an S-basis β_i for E_{λ_i} and $\coprod_{i=1}^m \beta_i$ is the required basis β. Indeed, $[S]_\beta$ is diagonal by construction of β_i and prop. 5.4(iii). $[T]_\beta$ is diagonal because E_{λ_i} are eigenspaces of T, for $i = 1, \dots, m$.

Characteristic polynomial

Def. For $A \in M_{n\times n}(\mathbb{R})$, the characteristic polynomial $\chi_A = \det(A - xI)$.

For $T: V \to V$ ", the characteristic polynomial $\chi_T = \det(A - xI)$ where $[T]_\beta = A$ for some basis.

Remark: 1). χ_T is independent of choice of basis, so $\chi_A = \chi_{PAP^{-1}}$.

Moreover, for any polynomial f, $f(PAP^{-1}) = P f(A) P^{-1}$. To see this, say $f = a_0 + a_1 x + \dots + a_n x^n$

$$\begin{aligned} f(PAP^{-1}) &= a_0 + a_1 PAP^{-1} + a_2 PAP^{-1}PAP^{-1} + \dots + a_n (PAP^{-1})^n \\ &= a_0 PP^{-1} + a_1 PAP^{-1} + a_2 PA^2P^{-1} + \dots + a_n PA^n P^{-1} \\ &= P(a_0 + a_1 A + a_2 A^2 + \dots + a_n A^n) P^{-1} = P f(A) P^{-1} \end{aligned}$$

In particular, $\chi_A(PAP^{-1}) = P\chi_A(A)P^{-1} = 0$ by Cayley-Hamilton. Also $\chi_{PAP^{-1}}(A) = 0$.

2). $\chi_A = \det(A - xI) = (-1)^n x^n + (-1)^{n-1} \underline{\operatorname{tr}(A)} x^{n-1} + \dots + \underline{\det A}$.

Invariant subspaces

Q6. Let $T: V \to V$ and $\dim V < \infty$. W is an invariant subspace under T. If $v_1, \dots, v_n$ are eigenvectors to T corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_n$ s.t. $v_1 + \dots + v_n \in W$, then $v_i \in W$ for all i.

pf. (Try to learn from the proof "$v_1, \dots, v_n$ are linearly independent").

First, we have

$$v_1 + \dots + v_n \in W \quad ①$$

As W is T-invariant, applying T to ① gives

$$T(v_1 + \dots + v_n) = Tv_1 + \dots + Tv_n = \lambda_1 v_1 + \dots + \lambda_n v_n \in W. \quad ②$$

Let $\lambda_1 \times ① - ②$ gives

$$(\lambda_1 - \lambda_2) v_2 + \dots + (\lambda_1 - \lambda_n) v_n \in W.$$

But then let $v_i' = (\lambda_1 - \lambda_i) v_i$.

$Tv_i' = T(\lambda_1 - \lambda_i) v_i = \lambda_i (\lambda_1 - \lambda_i) v_i = \lambda_i v_i'$.

so $v_2', \dots, v_n'$ are now in the situation of $n-1$ case. Induction gives the result. $\square$

Q7. Let $\mathbb{F} = \mathbb{C}$. Let $T: V \to V$ be a linear map. $\dim V < \infty$.

(i). Show that there exist $r \le \dim V$

$$\{0\} \subsetneq \ker T \subsetneq \ker T^2 \subsetneq \cdots \subsetneq \ker T^r = \ker T^{r+1} = \cdots$$

(ii) Show that

$$V \cong \ker T^r \oplus \underbrace{T^r V}_{\text{range } T^r}.$$

Now suppose the only eigenvalues for T are 0 and $\lambda \neq 0$. Let $W = \text{range } T^r$.

(iii). Show that W is a T-invariant and $T|_W$ has only eigenvalue λ. hint: use (ii) and Q4.

(iv). Let $S = (T - \lambda \mathrm{Id})|_W$. Show that 0 is the only eigenvalue of S and $S^m = 0$ for some m.

pf. (i) First, it is obvious $\ker T^i \subseteq \ker T^{i+1}$ as for any $v \in \ker T^i \Rightarrow T^i v = 0 \Rightarrow T^{i+1} v = 0 \Rightarrow v \in \ker T^{i+1}$

Second, we want to show there exist r s.t. $\ker T^r = \ker T^{r+1}$

Suppose not, $\{0\} \subsetneq \ker T \subsetneq \cdots \subsetneq \ker T^r \subsetneq \ker T^{r+1} \subsetneq \cdots$ continuous indefinitely, then $\exists\, x_i \in \ker T^i \setminus \ker T^{i-1}$ for all $i \in \mathbb{N}$. We claim $\{x_1, x_2, \dots\}$ is a linearly independent set.

Indeed, $x_i \notin \text{span}\{x_1, \dots, x_{i-1}\}$ as $x_i = \sum_{j=1}^{i-1} a_j x_j \Rightarrow T^{i-1} x_i = \sum_{j=1}^{i-1} a_j \underbrace{T^{i-1} x_j}_{0} = 0$ as $x_j \in \ker T^j \subseteq \ker T^{i-1}$ for all $j \le i-1$. $\Rightarrow x_i \in \ker T^{i-1} \Rightarrow \in$.

Hence $\{x_i\}_{i \in \mathbb{N}}$ is a linearly independent set in V. But V is finite dimensional, this is impossible, so there exist $r \le \dim V$, $\ker T^r = \ker T^{r+1}$.

Third, we show $\ker T^{r+k+1} = \ker T^{r+k}$ for all $k \in \mathbb{Z}_{\ge 0}$. Indeed, $v \in \ker T^{r+k+1} \Rightarrow T^{r+1}(T^k v) = 0$ $\Rightarrow T^k v \in \ker T^{r+1} = \ker T^r \Rightarrow T^r(T^k v) = 0 \Rightarrow v \in \ker T^{r+k}$

(ii). Pick a basis $\{e_i'\}$ for T^rV. Let $e_i \in V$ s.t. $T^r e_i = e_i'$. Then $e_i' \mapsto e_i$ extends to a linear map $\varphi: T^rV \longrightarrow V$, $e_i' \mapsto e_i$. As $T^r: V \to T^rV$, $e_i \mapsto e_i'$, we have $T^r \circ \varphi = id_{T^rV}$.

Claim: $V = \ker T^r \oplus \varphi(T^rV)$.

First, we show $V = \ker T^r + \varphi(T^rV)$.

Indeed, for any $v \in V$, $v = (v - \varphi(T^rv)) + \varphi T^r v$. It suffices to show $v - \varphi(T^rv) \in \ker T^r$.

But this is clear as $T^r(v - \varphi T^r v) = T^r v - \underbrace{T^r\varphi}_{id} T^r v = T^r v - T^r v = 0$.

Second, we show $\ker T^r \cap \varphi(T^rV) = \{0\}$.

For $v \in \varphi(T^rv) \cap \ker T^r$, say $v = \varphi T^r u$ for some $u \in V$. Then

$$\underset{\substack{\uparrow \\ v \in \ker T^r}}{0} = T^r v = \underbrace{T^r\varphi}_{id} T^r u = T^r u \underset{\substack{\uparrow \\ \text{apply } \varphi}}{\Rightarrow} 0 = \varphi(0) = \varphi T^r u = v. \Rightarrow v = 0.$$

This concludes the proof of claim, and hence (ii).

(iii). First, $T(W) \subseteq W$ as $T(T^rv) = T^r(\underbrace{Tv}_{\text{in } V}) \in T^rV = W$.

Second, for any T-eigenvector $w = T^ru$ ($\in W$) with eigenvalue $\alpha \in \{0, \lambda\}$. If $\alpha = 0$,

$T^{r+1}u = Tw = \alpha w = 0 \Rightarrow u \in \ker T^{r+1} = \ker T^r \Rightarrow w = T^r u = 0 \Rightarrow \Leftarrow$. So $\alpha = \lambda$.

(iv). As $T|_W$ has only eigenvalue λ, $S := (T - \lambda I)|_W$ has only eigenvalue 0.

Indeed, if $w = T^rv \in W$ is an eigenvector of S with eigenvalue α. Then

$$\alpha w = Sw = Tw - \lambda w. \Rightarrow Tw = (\lambda + \alpha) w.$$

By (iii), $\lambda + \alpha = \lambda$ so $\alpha = 0$.

For the claim $S^m = 0$ for some m, replacing V, T by W, S^m in (ii) we have

$$W = \ker S^m \oplus S^m W \quad \text{for some } m$$

As $\mathbb{F} = \mathbb{C}$, by Q4(i), if $S^m W \neq 0$, then there is a eigenvector $0 \neq w \in S^m W$.

But the only eigenvalue is zero, so $S^m w = 0$. Hence $w \in \ker S^m \cap S^m W = \{0\} \Rightarrow \Leftarrow$.

Hence $S^m W = 0$.

$\square$.

Q8. Let T be a linear operator on V. Suppose V is T-cyclic, i.e.,

$$V = \text{span}\{v, Tv, T^2v, \cdots\}$$

for some generator $v \in V$.

For another linear operator U on V, show that

$$TU = UT \iff U = g(T) \text{ for some polynomial } g(t).$$

proof. ($\Leftarrow$) is easy. As T commutes with any polynomial in T,

$$TU = Tg(T) = g(T)T = UT.$$

($\Rightarrow$). Let $v \in V$ be a generator of V. Then every $w \in V$ is written as

$$w = f(T)v \qquad \text{for some polynomial } f(t).$$

In particular, for $w = U(v)$,

$$U(v) = g(T)v \qquad \text{for some polynomial } g(t).$$

We claim $U = g(T)$. Indeed, for any $x \in V$, $x = h(T)v$ for some polynomial $h(t)$.

$$Ux = U(h(T)v) \underset{UT=TU}{=} h(T)Uv = h(T)g(T)v = g(T)h(T)v = g(T)x.$$

Since x is arbitrary, $U = g(T)$ $\square$.

<u>Cayley-Hamilton Theorem</u>

Q9. Let A be a 2×2 matrix with eigenvalue $-1, 2$. Find the inverse of $B = A - I$ in terms of A and I.

Ans. Since eigenvalue of A is $-1, 2$, eigenvalue of $B = A - I$ is $-2, 1$ as

$$Av = -v \Rightarrow (A-I)v = -v - v = -2v.$$

$$Av' = 2v' \Rightarrow (A-I)v' = 2v' - v' = v'.$$

Then $\chi_B = (x+2)(x-1)$. By Cayley-Hamilton theorem,

$$0 = (B+2I)(B-I) = B^2 + B - 2I$$

$$\Rightarrow B(B+I) = 2I. \Rightarrow B\left(\frac{A}{2}\right) = I$$

$$\Rightarrow B^{-1} = \frac{A}{2}.$$

$\square$.

Eigenspace and generalized eigenspace.

Def. 5.6. Let T be a linear operator on V and λ be an eigenvalue.

(i). The <u>eigenspace</u> for λ is

$$E_\lambda := \ker(T-\lambda I) = \{x \in V \mid Tx = \lambda x\}.$$

(ii). The <u>generalized eigenspace</u> for λ is

$$K_\lambda := \ker(T-\lambda I)^n = \{x \in V \mid (T-\lambda I)^n x = 0 \text{ for some } n\}.$$

for some $n \in \mathbb{N}$. If V is finite dimensional, we may take $n = \dim V$.

Remark. We have composite series

$$\{0\} \subsetneq \ker(T-\lambda I) \subsetneq \ker(T-\lambda I)^2 \subsetneq \cdots \subsetneq \ker(T-\lambda I)^r = \ker(T-\lambda I)^{r+1} = \cdots$$

where r is the first place the chain stablizes, and we have $r \le \dim V$.

In the definition we take $n = \dim V$ will be the biggest possible vector space of this form.

Eg. The most important example:

$$V = \{\mathbb{R} \longrightarrow \mathbb{C} \text{ differentiable}\} \qquad D = \frac{d}{dt} : V \to V \quad \text{is a linear operator.}$$

- For any $\lambda \in \mathbb{C}$, the λ-eigenspace is

$$E_\lambda = \{f \in V \mid \frac{df}{dt} = \lambda f\}$$

This is a first order differential equation.

$$E_\lambda = \{A e^{\lambda t} \mid A \in \mathbb{C}\}.$$

It is one dimensional, with geometric multiplicity 1.

- For the same λ, the generalized λ-eigenspace appear when you solve higher order DE's.

e.g. $$f'' - 2f' + f = 0$$

is written as $$(D-I)^2 f = 0$$

One solution is e^t, the other is te^t. This is because

$$D(te^t) = Dt(e^t) + t(De^t) = e^t + te^t.$$

$$\Rightarrow \quad (D-I)(te^t) = e^t \in \ker(D-I).$$

$$\Rightarrow \quad te^t \in \ker(D-I)^2.$$

Hence, the generalized λ-eigenspace for D is

$$K_\lambda = \{p(t)e^{\lambda t} \mid p \in \underset{\uparrow}{\mathbb{C}[t]}\}.$$

polynomial with coefficients in $\mathbb{C}$.

This is infinite dimensional, as $\mathbb{C}[t]$ is infinite dimensional. The geometric multiplicity for K_λ is infinite.

We want to study generalized eigenspace because for T not diagonalizable, we may not have enough eigenvector to form a basis for V. For example, $V = \mathbb{R}^2$.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \underset{\uparrow}{\chi_A} = (x-1)^2. \qquad E_1 = \ker(A-I) = \ker\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\{e_1\}.$$

characteristic polynomial.

but V is 2-dimensional! $\quad K_1 = \ker(A-I)^2 = \ker\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \ker\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\{e_1, e_2\}$

Now by considering generalized eigenspace, we have a eigenbasis for V.

In general, we have the following theorem.

(Primary decomposition theorem / spectral decomposition theorem).

Theorem 5.7. Let V be a vector space with $\dim V < \infty$. T be a linear operator on V. $\lambda_1, \cdots, \lambda_r$ are distinct eigenvalue of T. Then

$$V = \bigoplus_{i=1}^{r} K_{\lambda_i}$$

Moreover, each K_{λ_i} is T-invariant

- $\chi_T = ((x-\lambda_1)^{q_1} \cdots (x-\lambda_r)^{q_r}$ the characteristic polynomial.
- $\chi_{T|_{K_{\lambda_i}}} = (x-\lambda_i)^{q_i}$ for $i = 1, \cdots, r$, $\sum_{i=1}^{\lambda} q_i = \dim V$.
- (algebraic multiplicity of λ_i) $\overset{\text{def}}{=} q_i = \dim K_{\lambda_i}$.

<u>Eigendecomposition</u>. $T \in \mathcal{L}(V)$. V finite dimensional. $n = \dim V$.

Question: Is it possible to decompose V into T-eigenspaces?
Ans: Four cases ① -1 $\Rightarrow$ ① -2 $\Rightarrow$ ② $\Rightarrow$ ③.

① Best Case: $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_n}$ diagonalizable

① -1. λ_i's are distinct, with eigenvectors v_i, $i = 1, \dots, n$.

$\Leftrightarrow$ $\dim E_{\lambda_i} = 1$, and $E_{\lambda_i} = \operatorname{span}\{v_i\}$ and λ_i distinct.

$\Leftrightarrow \{v_i\}_{i=1}^{n}$ are linearly independent

$\Leftrightarrow \{v_i\}_{i=1}^{n}$ form an eigenbasis of V.

$\Leftrightarrow \chi_T(x) = c(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$ splits into <u>distinct</u> factors.

Eg. $T = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix}$. $V = \mathbb{R}^4$.

① -2. Some of λ_i are repeated, but we still have an eigen basis.

$\Leftrightarrow$ T is diagonalizable.

$\Leftrightarrow$ every algebraic multiplicity = geometric multiplicity, for all λ_i.

(Some of E_{λ_i} can be more than one dimensional.)

$\Leftrightarrow \chi_T(x) = c(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_n)^{m_n}$ splits but may have repeated factors

Eg. $T = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}$ $V = \mathbb{R}^5$. is ① -2 but not ① -1.

If $\mathbb{F} = \mathbb{C}$ is algebraically closed, we will have $\chi_T(x)$ splits

② Not so good: $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_m}$ λ_i distinct, K_{λ_i} is the <u>generalized eigenspaces</u>, $m \le n$.

$\Leftrightarrow \chi_T(x) = c(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_m)^{\lambda_m}$ splits but may have repeated factors.

- $\dim K_{\lambda_i} = m_i$ = algebraic multiplicity.
- $\dim E_{\lambda_i}$ = geometric multiplicity $\le \dim K_{\lambda_i}$ = algebraic multiplicity.

Eg. $T = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$
e_1 e_4 e_5

$\chi_T(x) = (x-0)^4(x-1)^2$

$\lambda_1 = 0$. $E_0 = \operatorname{span}\{e_1, e_4\}$. $\subseteq K_0 = \operatorname{span}\{e_1, e_2, e_3, e_4\}$

$\lambda_2 = 1$. $E_1 = \operatorname{span}\{e_5\}$. $\subseteq K_1 = \operatorname{span}\{e_5, e_6\}$.

is ② but not ① -2 or ① -1.

If $\mathbb{F}=\mathbb{C}$, we always have $\chi_T(x)$ splits.

Thm. Whenever $\chi_T(x)$ splits, there exist basis β s.t. $[T]_\beta$ is in <u>Jordan normal form</u>:

$$\text{Eg.}\quad [T]_\beta = \begin{pmatrix} \lambda_1 & 1 & & & & & & & & \\ & \lambda_1 & 1 & & & & & & & \\ & & \lambda_1 & & & & & & & \\ & & & \lambda_1 & 1 & & & & & \\ & & & & \lambda_1 & & & & & \\ & & & & & \lambda_2 & & & & \\ & & & & & & \lambda_2 & 1 & & \\ & & & & & & & \lambda_2 & 1 & \\ & & & & & & & & \lambda_2 & 1 \\ & & & & & & & & & \lambda_2 \end{pmatrix}$$

size 3

size 2

size 1

size 4.

each of blocks is called Jordan block.

$\dim E_{\lambda_i}$ = # Jordan blocks with eigenvalue λ_i = # lin indep eigenvectors of λ_i = geometric multiplicity

$\dim K_{\lambda_i} = \sum$ size of all Jordan blocks with eigenvalue λ_i = algebraic multiplicity.

= largest exponent m_i s.t. $(x-\lambda_i)^{m_i} \mid \chi_T(x)$.

Each Jordan block generates a T-cyclic space, for example,

$$T=\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$$ has generator e_3 as

$$\{e_3, \underset{\underset{e_2}{\parallel}}{Te_3}, \underset{\underset{e_1}{\parallel}}{T^2e_3}\} \text{ is a basis of } \mathbb{R}^3$$

③ Worst case. $\chi_T(x) = c\prod_{i=1}^{k} f_i^{q_i}$ f_i irreducible, but may not be linear, i.e., $\chi_T(x)$ may not splits.

Eg. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ when $\theta \notin k\pi\mathbb{Z}$. $\mathbb{F}=\mathbb{R}$. $V=\mathbb{R}^2$.

$\chi_T(x) = x^2 - 2\cos\theta\, x + 1$, discriminant $\Delta = 4(\cos^2\theta - 1) < 0$ for $\theta \notin \pi\mathbb{Z}$.

has no solution in $\mathbb{R}$! so this example is ③ but not ① or ②.

$K_\lambda \supseteq E_\lambda = \{0\}$ for all $\lambda \in \mathbb{R}$. Therefore, we cannot decompose $V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_m}$.

(*) A more general version of <u>primary decomposition</u> is available.

$$V = \bigoplus_{i=1}^{k} (\ker f_i)^{q_i}$$

Ref. "https://math.mit.edu/~dav/generalized.pdf" notes on "generalized eigenspaces", 2019 (Thm 6.7)
Artin, Algebra, Prentice Hall Inc, 1991

Eigen decomposition.

Let $T: V \to V$ be a linear operator. $\dim V = n$. How to decompose V into eigenspaces of T?

Tools / Descriptors.

① Characteristic polynomial
algebraic multiplicity.
generalized eigenspace.

② geometric multiplicity
eigenspaces.

② Primary decomposition – See tutorial notes (5-11)-(5-12).

Q. What does it mean to plug in T into a polynomial? (out of syllabus)

A: $\mathrm{End}(V) = \mathrm{Hom}(V,V) = \mathcal{L}(V)$ is a ring. Its product structure is given by composition.

$\mathbb{F}[t]$ is also a ring.

Fix a linear operator T, we have a canonical map.

$$\mathbb{F}[t] \to \mathrm{End}(V)$$
$$f(t) \mapsto f(T)$$

which extends by linearity from the maps $t \mapsto T$, $1 \to \mathrm{Id}$.

In particular, $0 \mapsto 0_V$, the zero map 0_V.