MATH2040 Midterm 2
Reference Solution

1. (40 pts) Determine whether the following statements are true or false. If it is true, prove it; if it is false, give a counterexample.
   (a) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.
   (b) A linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.
   (c) Let T be an invertible linear operator. A scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.
   (d) If T is a linear operator on a finite-dimensional vector space V, then for any $v \in V$ the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $T(v)$.

Solution:

(a) False.

Consider $T : \mathbb{R} \to \mathbb{R}$ be the identity map on $\mathbb{R}$. Then $1 \in \mathbb{R}$ is an eigenvalue of T but $2 = 1 + 1$ is not. (In fact, 1 is the unique eigenvalue of T)

This is Question 5.1.1(f) in Practice Problems of Homework 4.

(b) True.

Suppose T is invertible. Then for all $v \neq 0$ we have $Tv \neq 0 = 0 \cdot v$. This implies that 0 is not an eigenvalue of T.

Suppose T is not invertible. Since the vector space V is finite-dimensional, this implies that T is not one-to-one. So there exists nonzero $v \in \mathsf{N}(\,T\,)$, or equivalently $Tv = 0 = 0 \cdot v$. Hence 0 is an eigenvalue of T (with an eigenvector v).

Thus T is invertible if and only if 0 is not an eigenvalue of T.

This is Question 5.1.8(a) in Practice Problems of Homework 4.

(c) True.

Suppose λ is an eigenvalue of T. Since T is invertible, $\lambda \neq 0$. Also, by the definition of eigenvalue, there exists nonzero $v \in V$ such that $Tv = \lambda v$, so $T^{-1}v = T^{-1}(\lambda^{-1}\lambda v) = \lambda^{-1}T^{-1}(Tv) = \lambda^{-1}v$. Since v is nonzero, this implies that λ^{-1} is an eigenvalue of T^{-1}.

Suppose λ^{-1} is an eigenvalue of T^{-1}. Since T is invertible, T^{-1} is also invertible. So λ^{-1} is nonzero. By the previous proof, $\lambda = \left(\lambda^{-1}\right)^{-1}$ is an eigenvalue of $\left(T^{-1}\right)^{-1} = T$.

So λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

This is Question 5.1.8(b) in Practice Problems of Homework 4.

(d) False.

Consider $T = \mathsf{L}_A : \mathbb{R}^2 \to \mathbb{R}^2$ where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$. Observe that $T(v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T^2(v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So the T-cyclic subspace generated by v is $\mathrm{Span}(\ \{\ v, Tv\ \}\ ) = \mathbb{R}^2$ while the T-cyclic subspace generated by $T(v)$ is $\mathrm{Span}(\ \{\ Tv\ \}\ ) = \left\{\ \begin{pmatrix} a \\ 0 \end{pmatrix}\ :\ a \in \mathbb{R}\ \right\} \neq \mathbb{R}^2$.

This is Question 5.4.1(d) in Practice Problems of Homework 5.

2. (30 pts) Fo each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$

Solution:

(a) The characteristic polynomial of A is $p(t) = \det(A - tI_2) = t^2 - 2t - 8 = (t+2)(t-4)$. As it has $2 = \dim(\mathbb{R}^2)$ distinct roots, A is diagonalizable.

The eigenvalues of A are $-2, 4$.

- For $\lambda = -2$, the eigenspace is $E_{-2} = \mathsf{N}\,(\,A + 2I_2\,) = \mathsf{N}\left(\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\right) = \mathrm{Span}\left(\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}\right)$, with basis $\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$
- For $\lambda = 4$, the eigenspace is $E_4 = \mathsf{N}\,(\,A - 4I_2\,) = \mathsf{N}\left(\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\right) = \mathrm{Span}\left(\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}\right)$, with basis $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

As $\beta = \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ consists of eigenvectors of A with distinct eigenvalues and is of size $|\,\beta\,| = 2 = \dim(\mathbb{R}^2)$, it is an eigenbasis for $\mathbb{R}^2$. So for $Q = [\mathrm{Id}]_\beta^\alpha = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ with α being the standard basis of $\mathbb{R}^2$, we have $D = Q^{-1}AQ = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$, which is diagonal.

(b) The characteristic polynomial of A is $p(t) = \det(A - tI_2) = (t-1)^2$, so A has unique eigenvalue $\lambda = 1$.

On this eigenvalue $\lambda = 1$, the eigenspace is $E_1 = \mathsf{N}\,(\,A - I_2\,) = \mathsf{N}\left(\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}\right) = \mathrm{Span}\left(\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}\right)$, which is of dimension 1.

As the eigenvalue $\lambda = 1$ has algebraic multiplicity 2 but geometric multiplicity $1 \neq 2$, A is not diagonalizable.

(c) The characteristic polynomial of A is $p(t) = \det(A - tI_3) = -(t-3)(t+2)(t-4)$. As it has $3 = \dim(\mathbb{R}^3)$ distinct roots, A is diagonalizable.

The eigenvalues of A are $-2, 3, 4$.

- For $\lambda = -2$, the eigenspace is $E_{-2} = \mathsf{N}\,(\,A + 2I_3\,) = \mathrm{Span}\left(\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}\right)$, with basis $\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$
- For $\lambda = 3$, the eigenspace is $E_3 = \mathsf{N}\,(\,A - 3I_3\,) = \mathrm{Span}\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}\right)$, with basis $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$.
- For $\lambda = 4$, the eigenspace is $E_4 = \mathsf{N}\,(\,A - 4I_3\,) = \mathrm{Span}\left(\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}\right)$, with basis $\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$.

As $\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ consists of eigenvectors of A with distinct eigenvalues and is of size $|\,\beta\,| = 3 = \dim(\mathbb{R}^3)$, it is an eigenbasis for $\mathbb{R}^3$. So for $Q = [\mathrm{Id}]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ with α being the standard basis of $\mathbb{R}^3$, we have $D = Q^{-1}AQ = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, which is diagonal.

3. (30 pts) Let $T = L_A : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$, where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find all eigenvalues and eigenvectors of T.

(b) Find an ordered basis β of $\mathbb{R}^3$ so that

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Find a polynomial $g(t)$ such that $T^{-1} = g(T)$.

Solution:

(a) The characteristic polynomial of T is $p(t) = \det(A - tI_3) = -t^3 + 3t^2 - 3t + 1 = -(t-1)^3$, so T has unique eigenvalue $\lambda = 1$.

The corresponding eigenspace is $E_1 = \mathsf{N}\,(\,A - I_3\,) = \mathsf{N}\left(\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \mathrm{Span}\left(\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right)$.

So the eigenvectors of T are exactly the elements in $E_1 \setminus \{0\} = \left\{ \begin{pmatrix} a \\ a \\ b \end{pmatrix} \;:\; a, b \in \mathbb{R},\ (a,b) \neq (0,0) \right\}$.

(b) For a basis $\beta = \{\,v_1, v_2, v_3\,\}$ that gives the required matrix representation, we must have $Tv_1 = v_1$, $Tv_2 = v_1 + v_2$, $Tv_3 = v_3$, so $v_1, v_3 \in E_1 = \mathsf{N}\,(\,T - \mathrm{Id}\,)$ and $(T - \mathrm{Id})v_2 = v_1 \in E_1 \setminus \{0\}$, which implies that $\{\,v_1, v_3\,\}$ is a basis of E_1 and $v_2 \in \mathsf{N}\left(\,(T - \mathrm{Id})^2\,\right) \setminus E_1$ since $\dim(E_1) = 2$.

Note that $p(t)$ splits and T has unique eigenvalue $\lambda = 1$, so $K_1 = \mathbb{R}^3$, $\dim(K_1) = \dim(\mathbb{R}^3) = 1 + \dim(E_1)$, which implies that $\mathsf{N}\left(\,(T - \mathrm{Id})^2\,\right) = \mathbb{R}^3$ (which we can also verify directly). So to construct the required basis, we need to choose a vector $v_2 \in \mathbb{R}^3 \setminus E_1$. In particular, we may select $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then with $v_1 = (T - \mathrm{Id})v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we have $v_1, v_3 \in E_1$, and it is easy to see that $\beta = \{\,v_1, v_2, v_3\,\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a linearly independent set of size $|\,\beta\,| = 3 = \dim(\mathbb{R}^3)$, hence a basis of $\mathbb{R}^3$.

By the choice of β it is easy to see that $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(c) By Cayley–Hamilton theorem, $-T^3 + 3T^2 - 3T + \mathrm{Id} = 0$, so $\mathrm{Id} = T(T^2 - 3T + 3\mathrm{Id})$, which gives $T^{-1} = T^2 - 3T + 3\mathrm{Id} = g(T)$ with polynomial $g(t) = t^2 - 3t + 3$.

See also Question 5.4.18(b) in Homework 5.