**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH 2040B (Second Term, 2022-23)**
**Linear Algebra II**
**Midterm 1**
**Date: 22 February 2023**

**Instructions:**

- Answer all 4 questions. Total score: 100 pts. Time allowed: 45 minutes.
- Give adequate explanation and justification for all your calculations and observations and write your proof in a clear and rigorous way.

1. (30 pts) Determine whether the following subsets are subspaces. If it is a subspace, prove it and **compute its dimension**. If it is not a subspace, briefly explain why.
   (a) $S_1 := \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$ as a subset of $\mathbb{R}^3$.
   (b) $S_2 := \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 - 4a_2^2 - a_3^2 = 0\}$ as a subset of $\mathbb{R}^3$.
   (c) $S_3 := \{f(x) \in P_n(\mathbb{R}) : f(-x) = -f(x)\}$ as a subset of $P_n(\mathbb{R})$.
   (d) The subset $S_4$ of $M_{3\times 3}(\mathbb{R})$ consisting of all non-invertible matrices.

2. (20 pts) For a $3\times 3$ matrix $A \in M_{3\times 3}(\mathbb{R})$, let $a_{ij}$ denote the $(i,j)^{th}$-entry of $A$. Determine whether the following map $f$ is a linear transformation. If it is, **write down a basis** of the null space $f^{-1}(0) = \{A \in M_{3\times 3}(\mathbb{R}) : f(A) = 0\}$ and **compute its dimension**; if not, give reasons.
   (a) $f = \mathrm{tr} : M_{3\times 3}(\mathbb{R}) \to \mathbb{R}$, given by $f(A) = \sum_{i=1}^{3} a_{ii}$, summing over diagonal entries.
   (b) $f = \det : M_{3\times 3}(\mathbb{R}) \to \mathbb{R}$, given by $f(A) = \det(A)$, the determinant.

3. (30 pts) Let $V$ be a vector space with the ordered basis $\beta = \{v_1, v_2, \cdots, v_n\}$. Define $v_0 = 0$.
   (a) Prove that there exists a linear transformation $T : V \to V$ such that $T(v_j) = v_j - 2v_{j-1}$ for $j = 1, 2 \cdots n$.
   (b) Compute $[T]_\beta$.
   (c) Is $T$ invertible? Prove or disprove it.

4. (20 pts) For each of the following linear operators, find the matrix representation of the transformation with respect to the standard basis.
   (a) $T_1$ is the reflection of $\mathbb{R}^2$ about the line $y = 3x$.
   (b) $T_2$ is the reflection of $\mathbb{R}^3$ about the plane $3y - z = 0$.