# MATH2040 Homework 7
# Reference Solution

6.3.3(c). For the following inner product space $V$ and linear operator $T$ on $V$, evaluate $T^*$ at the given vector in $V$.

$$V = \mathsf{P}_1(\mathbb{R}) \text{ with } \langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t$$
$$T(f) = f' + 3f, \quad f(t) = 4 - 2t$$

**Solution:** Let $T^*f = a + bt$ with $a, b \in \mathbb{R}$. Then for all $g \in V$, $\langle Tg, 4-2t \rangle = \langle Tg, f \rangle = \langle g, T^*f \rangle = \langle g, a+bt \rangle$. In particular,

$$24 = \langle 3, 4-2t \rangle = \langle T1, 4-2t \rangle = \langle 1, a+bt \rangle = 2a$$
$$4 = \langle 1+3t, 4-2t \rangle = \langle Tt, 4-2t \rangle = \langle t, a+bt \rangle = \frac{2}{3}b$$

and so $a = 12$, $b = 6$, $T^*f = 12 + 6t$.

### Note

$T^*(a+bt) = 3a + 3(a+b)t$

6.3.13. Let $T$ be a linear operator on a finite-dimensional inner product space $V$. Prove the following results.

(a) $\mathsf{N}(T^*T) = \mathsf{N}(T)$. Deduce that $\operatorname{rank}(T^*T) = \operatorname{rank}(T)$

(b) $\operatorname{rank}(T) = \operatorname{rank}(T^*)$. Deduce from (a) that $\operatorname{rank}(TT^*) = \operatorname{rank}(T)$

(c) For any $n \times n$ matrix $A$, $\operatorname{rank}(A^*A) = \operatorname{rank}(AA^*) = \operatorname{rank}(A)$

**Solution:**

(a) For each $v \in \mathsf{N}(T)$ we have $Tv = 0$ and so $T^*Tv = T^*0 = 0$, $v \in \mathsf{N}(T^*T)$. This implies that $\mathsf{N}(T) \subseteq \mathsf{N}(T^*T)$.

For each $v \in \mathsf{N}(T^*T)$ we have $T^*Tv = 0$ and so $0 = \langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = \| Tv \|^2$, which implies that $Tv = 0$, $v \in \mathsf{N}(T)$. This implies that $\mathsf{N}(T^*T) \subseteq \mathsf{N}(T)$.

Hence $\mathsf{N}(T) = \mathsf{N}(T^*T)$.

Since $V$ is finite-dimensional, we have $\operatorname{rank}(T^*T) = \dim(V) - \dim \mathsf{N}(T^*T) = \dim(V) - \dim \mathsf{N}(T) = \operatorname{rank}(T)$.

(b) Let $\alpha$ be an orthonormal basis for $V$. Then $\operatorname{rank}(T) = \operatorname{rank}([T]_\alpha) = \operatorname{rank}([T]_\alpha^*) = \operatorname{rank}([T^*]_\alpha) = \operatorname{rank}(T^*)$ since by MATH1030 the column rank of a matrix equals its row rank.

Using part (a) on $T^*$ we have $\operatorname{rank}(TT^*) = \operatorname{rank}(T^*) = \operatorname{rank}(T)$.

### Note

You can also use Question 6.3.12 (in Practise Problem).

(c) By previous parts, $\operatorname{rank}(A^*A) = \operatorname{rank}(\mathsf{L}_{A^*A}) = \operatorname{rank}(\mathsf{L}_{A^*}\mathsf{L}_A) = \operatorname{rank}(\mathsf{L}_A^*\mathsf{L}_A) = \operatorname{rank}(\mathsf{L}_A) = \operatorname{rank}(A)$. Similarly, $\operatorname{rank}(AA^*) = \operatorname{rank}(A^*) = \operatorname{rank}(\mathsf{L}_A^*) = \operatorname{rank}(\mathsf{L}_A) = \operatorname{rank}(A)$. So $\operatorname{rank}(A^*A) = \operatorname{rank}(AA^*) = \operatorname{rank}(A)$.

6.3.14. Let $V$ be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y \rangle z$ for all $x \in V$. First prove that $T$ is linear. Then show that $T^*$ exists, and find an explicit expression for it.

**Solution:**

(a) Let $x, x' \in V$, $c \in \mathbb{F}$ be a scalar. Then $T(x+x') = \langle\, x+x',\, y \,\rangle z = (\langle\, x,\, y \,\rangle + \langle\, x',\, y \,\rangle)z = \langle\, x,\, y \,\rangle z + \langle\, x',\, y \,\rangle z = T(x) + T(x')$, $T(cx) = \langle\, cx,\, y \,\rangle z = c\langle\, x,\, y \,\rangle x = cT(x)$. As $x, x', c$ are arbitrary, $T$ is linear.

(b) We will show that $T^*$ exists by constructing it explicitly. Define $S : V \to V$ by $S(x') = \langle\, x',\, z \,\rangle y$ for all $x' \in V$. By the same argument as above, $S$ is linear. Also, for all $x, x' \in V$ we have $\langle\, x,\, Sx' \,\rangle = \langle\, x,\, \langle\, x',\, z \,\rangle y \,\rangle = \langle\, x,\, y \,\rangle \langle\, z,\, x' \,\rangle = \langle\, \langle\, x,\, y \,\rangle z,\, x' \,\rangle = \langle\, Tx,\, x' \,\rangle$. This implies that $T^*$ exists and $T^* = S$.

6.3.15. Let $T : V \to W$ be a linear transformation, where $V$ and $W$ are finite-dimensional inner product spaces with inner products $\langle\, \cdot, \cdot \,\rangle_1$ and $\langle\, \cdot, \cdot \,\rangle_2$ respectively. Prove the following results.

(a) There is a unique adjoint $T^*$ of $T$, and $T^*$ is linear

(b) If $\beta$ and $\gamma$ are orthonormal bases for $V$ and $W$, respectively, then $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$

(c) $\operatorname{rank}(T^*) = \operatorname{rank}(T)$

(d) $\langle\, T^*(x),\, y \,\rangle_1 = \langle\, x,\, T(y) \,\rangle_2$ for all $x \in W$ and $y \in V$

(e) For all $x \in V$, $T^*T(x) = 0$ if and only if $T(x) = 0$

**Solution:**

(a) Since $V$ is finite-dimensional, there exist integer $n \in \mathbb{N}$ and orthonormal basis $\beta = \{\, e_1, \ldots, e_n \,\}$ for $V$.

Define $T^* : W \to V$ by $T^*(w) = \sum_{i=1}^n \langle\, w,\, Te_i \,\rangle_2\, e_i$ for each $w \in W$. By the property of inner product, $T^*$ is linear. Then for all $v \in V$ and $w \in W$ we have $\langle\, v,\, T^*w \,\rangle_1 = \sum_{i=1}^n \langle\, v,\, e_i \,\rangle_1 \langle\, Te_i,\, w \,\rangle_2 = \langle\, T\sum_{i=1}^n \langle\, v,\, e_i \,\rangle_1 e_i,\, w \,\rangle_2 = \langle\, Tv,\, w \,\rangle_2$, so $T^*$ is an adjoint of $T$. In particular, there exists an adjoint $T^*$ of $T$. Moreover, $T^*$ is linear.

Let $S : V \to W$ be an adjoint of $T$. Then for all $v \in V$ and $w \in W$ we have $\langle\, v,\, T^*w \,\rangle_1 = \langle\, Tv,\, w \,\rangle_2 = \langle\, v,\, S(w) \,\rangle_1$, so $\langle\, v,\, T^*w - S(w) \,\rangle_1 = 0$. In particular, for all $w \in W$ we have $\|\, T^*w - S(w) \,\|_1^2 = \langle\, T^*w - S(w),\, T^*w - S(w) \,\rangle_1 = 0$, so $T^*w = S(w)$. This implies that $S = T^*$. As $S$ is arbitrary, the adjoint of $T$ is unique.

## Note

In particular, (the construction of) adjoint is independent of the orthonormal basis chosen.

(b) Assume that $\beta = \{\, e_1, \ldots, e_n \,\}$ and $\gamma = \{\, f_1, \ldots, f_m \,\}$ for some $n, m \in \mathbb{N}$. Then $([T^*]_\gamma^\beta)_{ij} = \langle\, T^*f_j,\, e_i \,\rangle_1 = \overline{\langle\, e_i,\, T^*f_j \,\rangle_1} = \overline{\langle\, Te_i,\, f_j \,\rangle_2} = \overline{([T]_\beta^\gamma)_{ji}} = (([T]_\beta^\gamma)^*)_{ij}$ for all $i \in \{\, 1, \ldots, n \,\}$ and $j \in \{\, 1, \ldots, m \,\}$. This implies that $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$.

(c) Let $\beta, \gamma$ be orthonormal basis for $V, W$ respectively. Then $\operatorname{rank}(T) = \operatorname{rank}([T]_\beta^\gamma) = \operatorname{rank}(([T]_\beta^\gamma)^*) = \operatorname{rank}([T^*]_\gamma^\beta) = \operatorname{rank}(T^*)$ since by MATH1030 the column rank of a matrix equals its row rank.

(d) For $x \in W$ and $y \in V$, we have $\langle\, T^*x,\, y \,\rangle_1 = \overline{\langle\, y,\, T^*x \,\rangle_1} = \overline{\langle\, Ty,\, x \,\rangle_2} = \langle\, x,\, Ty \,\rangle_2$.

(e) Let $x \in V$.

Suppose $Tx = 0$. Then $T^*T(x) = T^*(Tx) = T^*(0) = 0$.

Suppose $T^*T(x) = 0$. Then $0 = \langle\, x,\, T^*Tx \,\rangle_1 = \langle\, Tx,\, Tx \,\rangle_1 = \|\, Tx \,\|_1^2$, which implies that $Tx = 0$.

As $x$ is arbitrary, $Tx = 0$ if and only if $T^*Tx = 0$ for all $x \in V$.

## Note

This proposition extends the results for adjoint operators in $\mathcal{L}(V)$ to $\mathcal{L}(V, W)$.

6.4.2(d). For the linear operator $T$ on an inner product space $V$, determine whether $T$ is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of $T$ for $V$ and list the corresponding eigenvalues.

$$V = \mathsf{P}_2(\mathbb{R}) \text{ and } T \text{ is defined by } T(f) = f', \text{ where } \langle\, f,\, g \,\rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t$$

**Solution:** We will apply Gram–Schmidt process on the standard basis $\alpha = \left\{\, 1, t, t^2 \,\right\}$ to find an orthonormal basis for $V$.

- $v_1 = w_1 = 1$ with norm $\| w_1 \| = 1$, so $e_1 = w_1 / \| w_1 \| = 1$
- $v_2 = t$, $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\| w_1 \|^2} w_1 = t - \frac{1}{2}$ with norm $\| w_2 \| = \frac{1}{2\sqrt{3}}$, so $e_2 = w_2 / \| w_2 \| = 2\sqrt{3}(t - \frac{1}{2})$
- $v_3 = t^2$, $w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\| w_1 \|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\| w_2 \|^2} w_2 = t^2 - t + \frac{1}{6}$ with norm $\| w_3 \| = \frac{1}{6\sqrt{5}}$, so $e_3 = w_3 / \| w_3 \| = 6\sqrt{5}(t^2 - t + \frac{1}{6})$

so $\beta = \{ 1, 2\sqrt{3}(t - \frac{1}{2}), 6\sqrt{5}(t^2 - t + \frac{1}{6}) \}$ is an orthonormal basis for $V$.

The matrix representation of $T$ on $\beta$ is $[T]_\beta = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}$, so $[T^*]_\beta = [T]_\beta^* = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}$. Since $[TT^*]_\beta = [T]_\beta [T^*]_\beta = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 60 \end{pmatrix} = [T^*]_\beta [T]_\beta = [T^*T]_\beta$, $TT^* \neq T^*T$ and so $T$ is not normal. In particular, $T$ is not self-adjoint.

By spectral theorem, there is no orthonormal eigenbasis of $T$.

## Note

Note that it is not sufficient to check if $[T]_\alpha$ is a normal matrix (it is not). See also the operator in this question.

Adjoint of $T$ is easier to find if certain boundary condition is posed.

See also this and more generally this for studies on $\beta$.

$[T^*]_\alpha = \begin{pmatrix} -6 & 2 & 3 \\ 12 & -24 & -26 \\ 0 & 30 & 30 \end{pmatrix}$. See also this.

6.4.7. Let $T$ be a linear operator on an inner product space $V$, and let $W$ be a $T$-invariant subspace of $V$. Prove the following results.

(a) If $T$ is self-adjoint, then $T_W$ is self-adjoint

(b) $W^\perp$ is $T^*$-invariant

(c) If $W$ is both $T$- and $T^*$-invariant, then $(T_W)^* = (T^*)_W$

(d) If $W$ is both $T$- and $T^*$-invariant and $T$ is normal, then $T_W$ is normal

**Solution:**

(a) Suppose $T$ is self-adjoint. Then for all $w_1, w_2 \in W \subseteq V$ we have $\langle T_W w_1, w_2 \rangle = \langle T w_1, w_2 \rangle = \langle w_1, T w_2 \rangle = \langle w_1, T_W w_2 \rangle$. This implies that $T_W$ is also self-adjoint.

(b) Let $v \in W^\perp$. Then for all $w \in W$, $Tw \in W$ and so $0 = \langle Tw, v \rangle = \langle w, T^* v \rangle$. This implies that $T^* v \in W^\perp$. As $v$ is arbitrary, $W^\perp$ is $T^*$-invariant.

(c) Suppose $W$ is $T$- and $T^*$-invariant. Then $T_W$ and $(T^*)_W$ are both well-defined. For all $w_1, w_2 \in W$ we have $\langle T_W w_1, w_2 \rangle = \langle T w_1, w_2 \rangle = \langle w_1, T^* w_2 \rangle = \langle w_1, (T^*)_W w_2 \rangle$. This implies that $(T_W)^* = (T^*)_W$.

(d) Suppose $W$ is $T$- and $T^*$-invariant, and $T$ is normal. By the previous part, $(T_W)^* = (T^*)_W$, so $T_W (T_W)^* = T_W (T^*)_W = (TT^*)_W = (T^*T)_W = (T^*)_W T_W = (T_W)^* T_W$. This implies that $T_W$ is normal.

6.4.9. Let $T$ be a normal operator on a finite-dimensional inner product space $V$. Prove that $\mathsf{N}(T) = \mathsf{N}(T^*)$ and $\mathsf{R}(T) = \mathsf{R}(T^*)$

**Solution:** Since $T$ is normal, for all $v \in V$ we have $\| Tv \| = \| T^* v \|$ and thus $Tv = 0$ if and only if $T^* v = 0$. This implies $\mathsf{N}(T) = \mathsf{N}(T^*)$.

Trivially $\mathsf{R}(TT^*) \subseteq \mathsf{R}(T)$. By the result of Question 6.3.13(b) we have $\dim(\mathsf{R}(TT^*)) = \operatorname{rank}(TT^*) = \operatorname{rank}(T) = \dim(\mathsf{R}(T))$, so $\mathsf{R}(TT^*) = \mathsf{R}(T)$. Similarly $\mathsf{R}(T^*T) = \mathsf{R}(T^*)$. As $T$ is normal, $T^*T = TT^*$, so $\mathsf{R}(T) = \mathsf{R}(TT^*) = \mathsf{R}(T^*T) = \mathsf{R}(T^*)$.

**Note**

You can also use Question 6.3.12 (in Practise Problem) to work on the range.

6.4.10. Let $T$ be a self-adjoint operator on a finite-dimensional inner product space $V$. Prove that for all $x \in V$, $\| T(x) \pm ix \|^2 = \| T(x) \|^2 + \| x \|^2$. Deduce that $T - i\mathrm{Id}$ is invertible and that $\left[ (T - i\mathrm{Id})^{-1} \right]^* = (T + i\mathrm{Id})^{-1}$

**Solution:** Since $T$ is self-adjoint, we have $\langle Tx, x \rangle = \langle x, Tx \rangle$ for all $x \in V$. Then for each $x \in V$, we have $\| Tx \pm ix \|^2 = \| Tx \|^2 + \| x \|^2 \pm (\langle Tx, ix \rangle + \langle ix, Tx \rangle) = \| Tx \|^2 + \| x \|^2 \pm i(-\langle Tx, x \rangle + \langle x, Tx \rangle) = \| Tx \|^2 + \| x^2 \|$.

Let $x \in V$ be nonzero. Then $\| x \| > 0$, so $\| (T - i\mathrm{Id})x \|^2 = \| Tx \|^2 + \| x \|^2 \geq \| x \|^2 > 0$. This implies that $(T - i\mathrm{Id})x \neq 0$. As $x \neq 0$ is arbitrary, $T - i\mathrm{Id}$ is injective on a finite-dimensional space, hence invertible. Similarly, $T + i\mathrm{Id}$ is also invertible.

Since $T$ is self-adjoint, $(T + i\mathrm{Id})^* = T^* - i\mathrm{Id} = T - i\mathrm{Id}$. Then for each $x, y \in V$, we have $\left\langle (T - i\mathrm{Id})^{-1} x, y \right\rangle = \left\langle (T - i\mathrm{Id})^{-1}, (T + i\mathrm{Id})(T + i\mathrm{Id})^{-1} y \right\rangle = \left\langle (T + i\mathrm{Id})^*(T - i\mathrm{Id})^{-1}, (T + i\mathrm{Id})^{-1} y \right\rangle = \left\langle x, (T + i\mathrm{Id})^{-1} y \right\rangle$. This implies that $\left[ (T - i\mathrm{Id})^{-1} \right]^* = (T + i\mathrm{Id})^{-1}$.

**Note**

$\overline{(\frac{1}{x-i})} = \frac{1}{x+i}$ for $x \in \mathbb{R}$.

6.4.12. Let $T$ be a normal operator on a finite-dimensional real inner product space $V$ whose characteristic polynomial splits. Prove that $V$ has an orthonormal basis of eigenvectors of $T$. Hence prove that $T$ is self-adjoint.

**Solution:** Since the characteristic polynomial of $T$ splits on $\mathbb{R}$, $T$ has full eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $V = \bigoplus K_{\lambda_i}(T)$. Hence it suffices to show that $K_{\lambda_i}(T) = E_{\lambda_i}(T)$ for each $i$, since then by the normality of $T$ the orthonormal bases on $E_{\lambda_i}(T)$ union to an orthonormal eigenbasis of $T$.

Let $\lambda \in \mathbb{R}$ be an eigenvalue of $T$, and $v \in K_\lambda(T)$. Then for some $k \in \mathbb{N}$ we have $(T - \lambda\mathrm{Id})^k v = 0$. If $k \leq 1$ we have $v \in E_\lambda(T)$. Hence we may assume that $k \geq 2$. Then $(T - \lambda\mathrm{Id})^{k-1} v \in \mathsf{N}(T - \lambda\mathrm{Id})$. Since $T$ is normal, $T - \lambda\mathrm{Id}$ is also normal, and thus by the result of Question 6.4.9 we have $\mathsf{N}(T - \lambda I) = \mathsf{N}((T - \lambda\mathrm{Id})^*) = \mathsf{N}(T^* - \lambda\mathrm{Id})$. So $(T^* - \lambda\mathrm{Id})(T - \lambda\mathrm{Id})^{k-1} v = 0$, which gives $0 = \left\langle (T^* - \lambda\mathrm{Id})(T - \lambda\mathrm{Id})^{k-1} v, (T - \lambda\mathrm{Id})^{k-2} v \right\rangle = \left\| (T - \lambda\mathrm{Id})^{k-1} v \right\|^2$, so $(T - \lambda\mathrm{Id})^{k-1} v = 0$. Repeating this process gives $(T - \lambda\mathrm{Id})v = 0$ and so $v \in E_\lambda(T)$. As $v$ is arbitrary, $E_\lambda(T) = K_\lambda(T)$.

Therefore $V$ has an orthonormal eigenbasis of $T$. By real spectral theorem, $T$ is self-adjoint.

**Note**

The iteration stops at $k = 1$ as then we need a $(T - \lambda\mathrm{Id})^{-1}$ on the second component of the inner product to go further, but $T - \lambda\mathrm{Id}$ is not invertible for eivenvalue $\lambda$.

On complex spaces the characteristic polynomial always splits, in which case the argument above gives a (partial) proof of the complex spectral theorem.

You can also push the matrix to the complex field and work there, which is basically complexification of the space.

6.5.2(c). For the following matrix $A$, find an orthogonal or unitary matrix $P$ and a diagonal matrix $D$ such that $P^* A P = D$.

$$\begin{pmatrix} 2 & 3 - 3i \\ 3 + 3i & 5 \end{pmatrix}$$

**Solution:** The characteristic polynomial of $A$ is $p(t) = \det(A - tI) = t^2 - 7t - 8 = (t + 1)(t - 8)$, so the eigenvalues of $A$ are $-1, 8$.

- For $\lambda = -1$, the eigenspace is $E_{-1} = \mathsf{N}(A + I) = \mathrm{Span}\left( \left\{ \begin{pmatrix} -1 + i \\ 1 \end{pmatrix} \right\} \right)$ with basis $\left\{ \begin{pmatrix} -1 + i \\ 1 \end{pmatrix} \right\}$

- For $\lambda = 8$, the eigenspace is $E_8 = \mathsf{N}(\,A - 8I\,) = \operatorname{Span}\left(\left\{\begin{pmatrix}1-i\\2\end{pmatrix}\right\}\right)$ with basis $\left\{\begin{pmatrix}1-i\\2\end{pmatrix}\right\}$

So $\beta = \left\{\begin{pmatrix}-1+i\\1\end{pmatrix}, \begin{pmatrix}1-i\\2\end{pmatrix}\right\}$ is an eigenbasis of $A$.

To obtain an orthonormal basis, we apply Gram–Schmidt process on $\beta$. Since $A$ is Hermitian, it is self-adjoint and so normal, which implies that the eigenvectors of distinct eigenvalues are orthogonal. Hence it suffices to normalize the basis vectors.

- $e_1 = \begin{pmatrix}-1+i\\1\end{pmatrix} / \left\| \begin{pmatrix}-1+i\\1\end{pmatrix} \right\| = \frac{1}{\sqrt{3}}\begin{pmatrix}-1+i\\1\end{pmatrix}$
- $e_2 = \begin{pmatrix}1-i\\2\end{pmatrix} / \left\| \begin{pmatrix}1-i\\2\end{pmatrix} \right\| = \frac{1}{\sqrt{6}}\begin{pmatrix}1-i\\2\end{pmatrix}$.

Thus $\gamma = \left\{\frac{1}{\sqrt{3}}\begin{pmatrix}-1+i\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1-i\\2\end{pmatrix}\right\}$ is an orthonormal eigenbasis of $A$, which gives $P = \begin{pmatrix}(-1+i)/\sqrt{3} & (1-i)/\sqrt{6}\\ 1/\sqrt{3} & 2/\sqrt{6}\end{pmatrix}$ and $D = \begin{pmatrix}-1 & 0\\ 0 & 8\end{pmatrix}$.

6.5.6. Let $V$ be the inner product space of complex-valued continuous functions on $[0, 1]$ with the inner product $\langle\, f,\, g\,\rangle = \int_0^1 f(t)\overline{g(t)}\,\mathrm{d}t$. Let $h \in V$, and define $T : V \to V$ by $T(f) = hf$. Prove that $T$ is a unitary operator if and only if $|\,h(t)\,| = 1$ for $0 \le t \le 1$.

**Solution:** Suppose $|\,h(t)\,| = 1$ for all $t \in [0, 1]$. Then $1 = |\,h(t)\,|^2 = h(t)\overline{h(t)}$ for all $t \in [0, 1]$. So for all $f, g \in V$, $\langle\, Tf,\, Tg\,\rangle = \langle\, hf,\, hg\,\rangle = \int_0^1 h(t)f(t)\overline{h(t)g(t)}\,\mathrm{d}t = \int_0^1 f(t)\overline{g(t)}\,\mathrm{d}t = \langle\, f,\, g\,\rangle$. This implies that $T$ is unitary.

Suppose $T$ is unitary. Then for all $f, g \in V$ we have $\int_0^1 |\,h(t)\,|^2 f(t)\overline{g(t)}\,\mathrm{d}t = \langle\, Tf,\, Tg\,\rangle = \langle\, f,\, g\,\rangle = \int_0^1 f(t)\overline{g(t)}\,\mathrm{d}t$ and so $\int_0^1 (|\,h(t)\,|^2 - 1)f(t)\overline{g(t)}\,\mathrm{d}t = 0$. It is easy to see that $|\,h\,|^2 - 1, 1 \in V$, so $0 = \int_0^1 (|\,h(t)\,|^2 - 1)(|\,h(t)\,|^2 - 1)\overline{1}\,\mathrm{d}t = \int_0^1 (|\,h(t)\,|^2 - 1)^2\,\mathrm{d}t$. Since $(|\,h(t)\,|^2 - 1)^2 \ge 0$ for all $t \in [0, 1]$, we must have $(|\,h(t)\,|^2 - 1)^2 = 0$ and so $|\,h(t)\,| = 1$ for all $t \in [0, 1]$.

Hence $T$ is unitary if and only if $|\,h(t)\,| = 1$ for all $t \in [0, 1]$.

6.5.13. Suppose that $A$ and $B$ are diagonalizable matrices. Prove or disprove that $A$ is similar to $B$ if and only if $A$ and $B$ are unitarily equivalent.

**Solution:** The equivalence does not hold.

Consider $A = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $B = \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} \in M_{2\times 2}(\mathbb{C})$. Then $A, B$ are diagonalizable as they both have two distinct eigenvalues $0, 1$. Furthermore, $A$ is a diagonal matrix consisting of eigenvalues of $B$, so $A$ is similar to $B$. By spectral theorem, $B$ is unitarily equivalent to such matrix if and only if $B$ is normal. It is easy to verify that $B$ is not normal, so $A$ is not unitarily equivalent to $B$.

### Note

The other direction, by definition of unitary equivalence, holds trivially.

## Practice Problems

6.3.1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

(a) Every linear operator has an adjoint.

(b) Every linear operator on $V$ has the form $x \mapsto \langle\, x,\, y\,\rangle$ for some $y \in V$.

(c) For every linear operator $T$ on $V$ and every ordered basis $\beta$ for $V$, we have $[T^*]_\beta = ([T]_\beta)^*$

(d) The adjoint of a linear operator is unique.

(e) For any linear operators $T$ and $U$ and scalars $a$ and $b$, $(aT + bU)^* = aT^* + bU^*$

(f) For any $n \times n$ matrix $A$, we have $(\mathsf{L}_A)^* = \mathsf{L}_{A^*}$

(g) For any linear operator $T$, we have $(T^*)^* = T$.

**Solution:**

(a) True. Note that it is false without the finite-dimensional assumption.

(b) False. Note that it is still false without the finite-dimensional assumption, even if we consider only scalar valued linear operators (which would make the original proposition true).

(c) False. Analogous result exists, but requires adjustment for the lack of orthonormality. Note that the definition of such matrix depends on the convention of inner product used.

(d) True

(e) False

(f) True

(g) True

6.3.9. Prove that if $V = W \oplus W^\perp$ and $T$ is the projection on $W$ along $W^\perp$, then $T = T^*$.

**Solution:** Let $x, y \in V$. Then $Tx, Ty \in W$ and $x - Tx, y - Ty \in W^\perp$ and thus $\langle\, Tx,\, y - Ty \,\rangle = \langle\, x - Tx,\, Ty \,\rangle = 0$, which implies that $\langle\, Tx,\, y \,\rangle = \langle\, Tx,\, Ty \,\rangle = \langle\, x,\, Ty \,\rangle$. As $x, y$ are arbitrary, $T^* = T$.

6.3.10. Let $T$ be a linear operator on an inner product space $V$. Prove that $\|\, T(x) \,\| = \|\, x \,\|$ for all $x \in V$ if and only if $\langle\, T(x),\, T(y) \,\rangle = \langle\, x,\, y \,\rangle$ for all $x, y \in V$.

**Solution:** We will assume that the scalar field is real. For the complex case, the proof is similar.

Suppose $\langle\, Tx,\, Ty \,\rangle = \langle\, x,\, y \,\rangle$ fo all $x, y \in V$. Then $\|\, Tx \,\|^2 = \langle\, Tx,\, Tx \,\rangle = \langle\, x,\, x \,\rangle = \|\, x \,\|^2$ and so $\|\, Tx \,\| = \|\, x \,\|$ for all $x \in V$.

Suppose $\|\, Tx \,\| = \|\, x \,\|$ for all $x \in V$. By the polar identity (Q6.1.20 in Homework 6), we have $\langle\, x,\, y \,\rangle = \frac{1}{4}(\|\, x + y \,\|^2 - \|\, x - y \,\|^2)$. So $\langle\, Tx,\, Ty \,\rangle = \frac{1}{4}(\|\, Tx + Ty \,\|^2 - \|\, Tx - Ty \,\|^2) = \frac{1}{4}(\|\, x + y \,\|^2 - \|\, x - y \,\|^2) = \langle\, x,\, y \,\rangle$ for all $x, y \in V$.

Thus $\|\, Tx \,\| = \|\, x \,\|$ for all $x \in V$ if and only if $\langle\, Tx,\, Ty \,\rangle = \langle\, x,\, y \,\rangle$ for all $x, y \in V$.

6.3.11. For a linear operator $T$ on an inner product space $V$, prove that $T^*T = T_0$ implies $T = T_0$. Is the same result true if we assume that $TT^* = T_0$?

**Solution:** By the result of Question 6.3.13(a), $V = \mathsf{N}(\, T_0 \,) = \mathsf{N}(\, T^*T \,) = \mathsf{N}(\, T \,)$. This implies that $T = T_0$.

Suppose $TT^* = T_0$. Since $(T^*)^* = T$, applying the previous result on $T^*$ we obtain $T^* = T_0$. This implies that $T = T_0^* = T_0$.

6.3.12. Let $V$ be an inner product space, and let $T$ be a linear operator on $V$. Prove that following results.

(a) $\mathsf{R}(\, T^* \,)^\perp = \mathsf{N}(\, T \,)$

(b) If $V$ is finite-dimensional, then $\mathsf{R}(\, T^* \,) = \mathsf{N}(\, T \,)^\perp$.

**Solution:**

(a) Let $v \in \mathsf{N}(\, T \,)$. Then $Tv = 0$. Then for all $w \in V$ we have $0 = \langle\, Tv,\, w \,\rangle = \langle\, v,\, T^*w \,\rangle$. This implies that $v \in \mathsf{R}(\, T^* \,)^\perp$.

Let $v \in \mathsf{R}(\, T^* \,)^\perp$. Then for all $w \in V$, $\langle\, Tv,\, w \,\rangle = \langle\, v,\, T^*w \,\rangle = 0$. In particular, $\langle\, Tv,\, Tv \,\rangle = 0$, so $Tv = 0$, $v \in \mathsf{N}(\, T \,)$.

Hence $\mathsf{N}(\, T \,) = \mathsf{R}(\, T^* \,)^\perp$.

(b) By the result of Question 6.2.13(c) in Homework 6, $\mathsf{N}(\, T \,)^\perp = (\mathsf{R}(\, T^* \,)^\perp)^\perp = \mathsf{R}(\, T^* \,)$.

**Note**

See also this, which uses Question 6.2.6 in Homework 6 instead of going through part (a).

Recall that the column rank of a matrix equals its row rank, and note how Question 6.3.13(b), 6.3.15(c), and 6.4.9 can also be done with the result of this question. See also this.

Counterexample for infinite-dimensional: $V = \ell^2(\mathbb{R})$ is the space of real square-summable sequences equipped with the usual inner product $\langle (a_n), (b_n) \rangle = \sum_{i=1}^{\infty} a_n b_n$, $T : V \to V$ is defined by $T((a_n)) = (\frac{1}{n} a_n)$.

6.4.1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

(a) Every self-adjoint operator is normal.

(b) Operators and their adjoints have the same eigenvectors.

(c) If $T$ is an operator on an inner product space $V$, then $T$ is normal if and only if $[T]_\beta$ is normal, where $\beta$ is any ordered basis for $V$.

(d) A real or complex matrix $A$ is normal if and only if $\mathsf{L}_A$ is normal.

(e) The eigenvalues of a self-adjoint operator must all be real.

(f) The identity and zero operators are self-adjoint.

(g) Every normal operator is diagonalizable.

(h) Every self-adjoint operator is diagonalizable.

**Solution:**

(a) True

(b) False

(c) False

(d) True

(e) True

(f) True

(g) False

(h) True

6.4.4. Let $T$ and $U$ be self-adjoint operators on an inner product space $V$. Prove that $TU$ is self-adjoint if and only if $TU = UT$.

**Solution:** The adjoint of $TU$ is $(TU)^* = U^*T^* = UT$. So $TU$ is self-adjoint if and only if $TU = UT$.

6.4.8. Let $T$ be a normal operator on a finite-dimensional complex inner product space $V$, and let $W$ be a subspace of $V$. Prove that if $W$ is $T$-invariant, then $W$ is also $T^*$-invariant.

**Solution:** If $W$ is trivial, then the proposition is also trivial. Hence we will assume that $W$ is nontrivial.

By the result of Question 5.4.24 in Homework 5, $T_W$ is diagonalizable, and $W = \bigoplus_{\lambda \in S} W \cap E_\lambda(T)$ for some (nonempty) set $S \subseteq \mathbb{C}$ of eigenvalues of $T$. Since $T$ is normal, $E_\lambda(T) = E_{\overline{\lambda}}(T^*)$. This implies that $W = \bigoplus_{\lambda \in S} W \cap E_{\overline{\lambda}}(T^*)$. Since $W \cap E_{\overline{\lambda}}(T^*)$ is a subspace of $V$ for all $\lambda \in S$, each of $W \cap E_{\overline{\lambda}}(T^*)$ is $T^*$-invariant. This implies that $W$ is also $T^*$-invariant.

6.4.14. Let $V$ be a finite-dimensional real inner product space, and let $U$ and $T$ be self-adjoint linear operators on $V$ such that $UT = TU$. Prove that there exists an orthonormal basis for $V$ consisting of vectors that are eigenvectors of both $U$ and $T$.

**Solution:** We will show this proposition by induction on the dimension of $V$. Trivially the proposition holds if $\dim(V) \leq 1$.

Suppose for some integer $n \in \mathbb{Z}^+$ the proposition holds on all spaces of dimension less than $n$. Let $V$ be a real inner product space of dimension $\dim(V) = n$, and $U, T$ are commuting self-adjoint operators on $V$. Since $U, T$ are self-adjoint, by spectral theorem $U, T$ each has an orthonormal eigenbasis for $V$. Since $n \geq 1$, $T$ has an eigenvalue $\lambda \in \mathbb{R}$. Let the corresponding eigenspace be $W = E_\lambda(T)$. If $W = V$, then the proposition is trivial as every orthonormal eigenbasis of $U$ (which exists due to spectral theorem) is also an orthonormal eigenbasis of $T$. Hence we may assume that $W$ is proper.

Trivially $W$ is $T$-invariant. For each $w \in W$ we have $Tw = \lambda w$, so $T(Uw) = UTw = \lambda Uw$, $Uw \in W$. This implies that $W$ is also $U$-invariant. By the result of Question 6.4.7(c), $W^\perp$ is $T^* = T$- and $U^* = U$-invariant. This implies that $T_W, U_W, T_{W^\perp}, U_{W^\perp}$ are all well-defined. By the result of Question 6.4.7(a), all these operators are self-adjoint. Since $\dim(V) = \dim(W) + \dim(W^\perp)$ and $W$ is nontrivial proper, so is $W^\perp$. Hence $n = \dim(V) > \dim(W)$ and $n = \dim(V) > \dim(W^\perp)$. By induction assumption, there exists orthonormal basis $\gamma_W$ (resp. $\gamma_{W^\perp}$) for $W$ (resp. $W^\perp$) which consist of eigenvectors of both $U_W$ and $T_W$ (resp. $U_{W^\perp}$ and $T_{W^\perp}$), and thus consist of eigenvectors of both $U$ and $T$. Since $V$ is finite-dimensional, $V = W \oplus W^\perp$, and so $\gamma$ is an orthonormal basis for $V$.

By induction the proposition holds for all finite-dimensional real inner product spaces.

## Note

You can also decompose the whole space and use the same approach as Question 5.4.25 in Homework 5, which requires only minimal modifications.

6.5.1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

(a) Every unitary operator is normal.

(b) Every orthogonal operator is diagonalizable.

(c) A matrix is unitary if and only if it is invertible.

(d) If two matrices are unitarily equivalent, then they are also similar.

(e) The sum of unitary matrices is unitary.

(f) The adjoint of a unitary operator is unitary.

(g) If $T$ is an orthogonal operator on $V$, then $[T]_\beta$ is an orthogonal matrix for any ordered basis $\beta$ for $V$.

(h) If all the eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.

(i) A linear operator may preserve the norm, but not the inner product.

**Solution:**

(a) True

(b) False

(c) False

(d) True

(e) False

(f) True

(g) False

(h) False

(i) False. See Question 6.3.10.

6.5.7. Prove that if $T$ is a unitary operator on a finite-dimensional inner product space $V$, then $T$ has a unitary square root: that is, there exists a unitary operator $U$ such that $T = U^2$.

**Solution:** Let $T = \sum_{i=1}^k \lambda_i T_i$ be the spectral decomposition of $T$, where $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ are the eigenvalues of $T$ and $T_1, \ldots, T_k$ are the corresponding orthogonal projection onto the eigenspaces. Let $\mu_1, \ldots, \mu_k \in \mathbb{C}$ be such that $\mu_i^2 = \lambda_i$ for each $i$. Then $U = \sum_{i=1}^k \mu_i T_i$ has the required properties:

- $U$ is trivially a linear operator on $V$
- $U^2 = \sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\mu_j T_iT_j = \sum_{i=1}^{k}\mu_i^2 T_i = T$
- $U^* = \sum_{i=1}^{k}\overline{\mu_i}T_i^* = \sum_{i=1}^{k}\overline{\mu_i}T_i$, so $U^*U = \sum_{i=1}^{k}\sum_{j=1}^{k}\overline{\mu_i}\mu_j T_iT_j = \sum_{i=1}^{k}\mid \mu_i \mid^2 T_i = \sum_{i=1}^{k}\mid \lambda_i \mid T_i = \sum_{i=1}^{k} T_i = \mathrm{Id}$ as $T$ is unitary and so $\mid \lambda_i \mid = 1$ for each $i$. Similarly $UU^* = \mathrm{Id}$. This implies that $U$ is unitary.

## Note

If $z \in \mathbb{C}$ with $\mid z \mid = 1$, then there exists $u \in \mathbb{C}$ also with $\mid u \mid = 1$ such that $u^2 = z$.

In general, if $p \in \mathsf{P}(\mathbb{C})$ is a nonconstant polynomial, then for every normal operator $T$ there exists a normal operator $U$ such that $p(U) = T$. Compare this with the fundamental theorem of algebra.

6.5.12. Let $A$ be an $n \times n$ real symmetric or complex normal matrix. Prove that $\det(A) = \prod_{i=1}^{n}\lambda_i$ where $\lambda_i$'s are the (not necessarily distinct) eigenvalues of $A$.

**Solution:** By assumption on $A$, there exists an orthonormal eigenbasis $\beta = \{ e_1, \ldots, e_n \}$ for $A$, with $\lambda_1, \ldots, \lambda_n$ being the corresponding eigenvalues. Then $A = Q\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,Q^*$ where $Q = [\mathrm{Id}]_\beta^\alpha$ and $\alpha$ is the standard basis of $\mathbb{F}^n$. So $\det(A) = \det(Q\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,Q^*) = \det(Q)\det(\mathrm{diag}(\lambda_1, \ldots, \lambda_n))\det(Q^*) = \prod_{i=1}^{n}\lambda_i$.

6.5.15. Let $U$ be a unitary operator on an inner product space $V$, and let $W$ be a finite-dimensional $U$-invariant subspace of $V$. Prove that

(a) $U(W) = W$

(b) $W^\perp$ is $U$-invariant.

**Solution:**

(a) Since $W$ is $U$-invariant, $U_W$ is well-defined. As $U$ is unitary, $U$ is injective, so $U_W$ is also injective. As $W$ is finite-dimensional, $U_W$ is bijective. Hence $U(W) = U_W(W) = W$.

(b) Let $v \in W^\perp$. Then $\langle v, w \rangle = 0$ for all $w \in W$. As $W$ is $U$-invariant, we have $\langle Uv, w \rangle = \left\langle Uv, U(U_W)^{-1}w \right\rangle = \left\langle v, (U_W)^{-1}w \right\rangle = 0$ for all $w \in W$ as $(U_W)^{-1}w \in W$. This implies that $Uv \in W^\perp$. As $v$ is arbitrary, $W^\perp$ is $U$-invariant.

6.5.16. Find an example of a unitary operator $U$ on an inner product space and a $U$-invariant subspace $W$ such that $W^\perp$ is not $U$-invariant.

**Solution:** Let $V = \ell^2(\mathbb{C})$ be the space of complex square-summable sequences, equipped with the (usual) inner product $\langle (a_n), (b_n) \rangle = \sum_{k=1}^{\infty} a_k\overline{b_k}$, and $U : V \to V$ be the right shift operator, $W = \{ (a_n) \in V \;:\; a_0 = 0 \}$. It is easy to see that $U$ is unitary and $W$ is $U$-invariant, but $W^\perp$ is not.

6.5.17. Prove that a matrix that is both unitary and upper triangular must be a diagonal matrix.

**Solution:** We will use induction on the size of the matrix. Trivially the proposition holds for $1 \times 1$ matrices.

Suppose the proposition holds for all $k \times k$ matrices for some $k \in \mathbb{Z}^+$. Let $A \in M_{(k+1)\times(k+1)}(\mathbb{F})$ be unitary and upper triangular. By assumption, $A = \begin{pmatrix} \lambda & C^* \\ 0_{k\times 1} & B \end{pmatrix}$ for some $\lambda \in \mathbb{F}$, $C \in M_{k\times 1}(\mathbb{F})$ and $B \in M_{k\times k}(\mathbb{F})$ with $B$ also being upper triangular. Then $A^* = \begin{pmatrix} \lambda^* & 0_{1\times k} \\ C & B^* \end{pmatrix}$ and so $A^*A = \begin{pmatrix} \lambda\lambda^* & \lambda^* C^* \\ \lambda C & CC^* + B^*B \end{pmatrix} = AA^* = \begin{pmatrix} \lambda\lambda^* + C^*C & C^*B^* \\ BC & BB^* \end{pmatrix} = I_{k+1} = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & I_k \end{pmatrix}$. This implies that

- $\lambda\lambda^* = \lambda\lambda^* + C^*C = 1$

- $\lambda C = BC = 0_{k\times 1}$
- $CC^* + B^*B = BB^* = I_k$

The first equality implies that $\lambda \neq 0$. The second equality implies that $C = 0_{k\times 1}$. This last equality implies that $B^*B = BB^* = I_k$. Hence $A = \begin{pmatrix} \lambda & 0_{1\times k} \\ 0_{k\times 1} & B \end{pmatrix}$ with $B \in M_{k\times k}(\mathbb{F})$ upper triangular and unitary. By induction assumption, $B$ is diagonal, so $A$ is also diagonal.
By induction, all unitary and upper triangular matrices are diagonal.

## Note

You can also use induction on columns, or induction on the ranges.