MATH2040 Homework 6
Reference Solution

6.1.8. Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a,b), (c,d) \rangle = ac - bd$ on $\mathbb{R}^2$

(b) $\langle A, B \rangle = \operatorname{tr}(A+B)$ on $M_{2\times 2}(\mathbb{R})$

(c) $\langle f, g \rangle = \int_0^1 f'(t)g(t)\,\mathrm{d}t$ on $\mathsf{P}(\mathbb{R})$ where $'$ denotes differentiation

Solution:

(a) Let $v = (1,1) \in \mathbb{R}^2$. Then $v \neq 0$ but $\langle v, v \rangle = 1 \cdot 1 - 1 \cdot 1 = 0$. So $\langle \cdot, \cdot \rangle$ is not an inner product on $\mathbb{R}^2$.

(b) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Then $A \neq 0_{2\times 2}$, but $\langle A, A \rangle = \operatorname{tr}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0$. So $\langle \cdot, \cdot \rangle$ is not an inner product on $M_{2\times 2}(\mathbb{R})$.

(c) Let $f(x) = 1$. Then $f \in \mathsf{P}(\mathbb{R})$ is nonzero, but $\langle f, f \rangle = \int_0^1 0 \cdot 1\,\mathrm{d}t = 0$. So $\langle \cdot, \cdot \rangle$ is not an inner product on $\mathsf{P}(\mathbb{R})$.

6.1.17. Let T he a linear operator on an inner product space V, and suppose that $\| T(x) \| = \| x \|$ for all x. Prove that T is one-to-one.

Solution: For all $x \in \mathsf{N}(T)$, we have $T(x) = 0$, so $\| x \| = \| T(x) \| = \| 0 \| = 0$, which by the property of norm implies $x = 0$. This implies that $\mathsf{N}(T) = \{0\}$, and so T is injective.

6.1.18. Let V be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and let W be an inner product space over $\mathbb{F}$ with inner product $\langle \cdot, \cdot \rangle$. If $T : V \to W$ is linear, prove that $\langle x, y \rangle' = \langle T(x), T(y) \rangle$ defines an inner product on V if and only if T is one-to-one.

Solution: For arbitrary $x, y, z \in V$ and $c \in \mathbb{F}$, we have

- $\langle x+y, z \rangle' = \langle T(x+y), T(z) \rangle = \langle T(x), T(z) \rangle + \langle T(y), T(z) \rangle = \langle x, z \rangle' + \langle y, z \rangle'$
- $\langle cx, y \rangle' = \langle T(cx), T(y) \rangle = c \langle T(x), T(y) \rangle = c \langle x, y \rangle'$
- $\overline{\langle y, x \rangle'} = \overline{\langle T(y), T(x) \rangle} = \langle T(x), T(y) \rangle = \langle x, y \rangle'$

Furthermore, for all $x \in V$, $\langle x, x \rangle' = \langle T(x), T(x) \rangle = \| T(x) \|^2$. Thus $\langle \cdot, \cdot \rangle'$ defines an inner product if and only if $\| T(x) \|^2 > 0$ for all $x \neq 0$, which holds if and only if $T(x) \neq 0$ for all $x \neq 0$, which holds if and only if T is injective.

6.1.19. Let V be an inner product space. Prove that

(a) $\| x \pm y \|^2 = \| x \|^2 \pm 2\Re \langle x, y \rangle + \| y \|^2$ for all $x, y \in V$, where $\Re \langle x, y \rangle$ denotes the real part of the complex number $\langle x, y \rangle$

(b) $| \| x \| - \| y \| | \leq \| x - y \|$ for all $x, y \in V$

Solution: For arbitrary $x, y \in V$,

(a)

$$\begin{aligned} \| x \pm y \|^2 &= \langle x \pm y, x \pm y \rangle \\ &= \langle x, x \rangle \pm \langle x, y \rangle \pm \langle y, x \rangle + \langle y, y \rangle \\ &= \| x \|^2 \pm (\langle x, y \rangle + \overline{\langle x, y \rangle}) + \| y \|^2 \\ &= \| x \|^2 \pm 2\Re \langle x, y \rangle + \| y \|^2 \end{aligned}$$

(b) By triangular inequality for norm, we have

$$\| x \| = \| (x-y)+y \| \le \| x-y \| + \| y \|$$
$$\| y \| = \| (y-x)+x \| \le \| x-y \| + \| x \|$$

Hence $-\| x-y \| \le \| x \| - \| y \| \le \| x-y \|$, or $| \, \| x \| - \| y \| \, | \le \| x-y \|$

6.1.23. Let $V = \mathbb{F}^n$, and let $A \in M_{n\times n}(\mathbb{F})$

(a) Prove that $\langle x, Ay \rangle = \langle A^*x, y \rangle$ for all $x, y \in V$

(b) Suppose that for some $B \in M_{n\times n}(\mathbb{F})$, we have $\langle x, Ay \rangle = \langle Bx, y \rangle$ for all $x, y \in V$. Prove that $B = A^*$

(c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

(d) Define linear operator T and U by $T(x) = Ax$ and $U(x) = A^*x$. Show that $[U]_\beta = [T]_\beta^*$ for any orthonormal basis β for V.

Solution:

(a) By the definition of the inner product on $V = \mathbb{F}^n$, we have $\langle x, Ay \rangle = \sum_{i=1}^n \overline{(Ay)_i} x_i = \sum_{i=1}^n \sum_{j=1}^n x_i \overline{A_{ij}} \overline{y_j} = \sum_{j=1}^n \sum_{i=1}^n (A^*)_{ji} x_i \overline{y_j} = \sum_{j=1}^n (A^*x)_j \overline{y_j} = \langle A^*x, y \rangle$ for all $x, y \in V$.

(b) Let $x, y \in V$. By part (a), $\langle Bx, y \rangle = \langle x, Ay \rangle = \langle A^*x, y \rangle$, so $\langle (B-A^*)x, y \rangle = 0$. Since y is arbitrary, we have $\langle (B-A^*)x, (B-A^*)x \rangle = 0$ and thus $(B-A^*)x = 0$. Since x is arbitrary, $B - A^* = 0_{n\times n}$, or $B = A^*$.

(c) By assumption, $\alpha = \{ e_1, \ldots, e_n \}$ and $\beta = \{ v_1, \ldots, v_n \}$ which are both orthonormal basis. By the definition of Q we have $Qe_i = v_i$ for all $i \in \{ 1, \ldots, n \}$. So for all $x = \sum_{i=1}^n x_i e_i, y = \sum_{i=1}^n y_i e_i \in V$ we have $\langle Qx, Qy \rangle = \sum_{i=1}^n \sum_{j=1}^n x_i \overline{y_j} \langle Qe_i, Qe_j \rangle = \sum_{i=1}^n \sum_{j=1}^n x_i \overline{y_j} \langle v_i, v_j \rangle = \langle x, y \rangle$.

For all $x, y \in V$ we have $\langle x, Qy \rangle = \left\langle Q(Q^{-1}x), Qy \right\rangle = \left\langle Q^{-1}x, y \right\rangle$. By part (b), $Q^* = Q^{-1}$.

Note

Also, $\delta_{ij} = \langle v_j, v_i \rangle = \langle Qe_j, Qe_i \rangle = \sum_{k=1}^n \sum_{l=1}^n \sum_{r=1}^n Q_{kl}\delta_{jl}(Q_{kr}\delta_{ir})^* = \sum_{k=1}^n (Q^*)_{ik} Q_{kj} = (Q^*Q)_{ij}$.

(d) Let β be an orthonormal basis for V. Let $Q \in M_{n\times n}(\mathbb{F})$ be defined as in part (c). Then $Q = [\mathrm{Id}]_\beta^\alpha$. By definition, $[T]_\alpha = A$ and $[U]_\alpha = A^*$, so $[U]_\beta = [\mathrm{Id}]_\alpha^\beta [U]_\alpha [\mathrm{Id}]_\beta^\alpha = Q^{-1}[T]_\alpha^* Q = (Q^*[T]_\alpha (Q^{-1})^*)^* = (Q^{-1}[T]_\alpha Q)^* = [T]_\beta^*$.

As β is arbitrary, $[U]_\beta = [T]_\beta^*$ for all orthonormal basis β for V.

6.1.29. Let V be a vector space over $\mathbb{C}$, and suppose that $[\,\cdot, \cdot\,]$ is a real inner product on V, where V is regarded as a vector space over $\mathbb{R}$, such that $[\, x, ix \,] = 0$ for all $x \in V$. Let $\langle \cdot, \cdot \rangle$ be the complex-valued function defined by

$$\langle x, y \rangle = [\, x, y \,] + i [\, x, iy \,] \quad \text{for } x, y \in V$$

Prove that $\langle \cdot, \cdot \rangle$ is a complex inner product on V.

Solution: Let $x, y, z \in V$, $c = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$. Then $0 = [\, x+y, i(x+y) \,] = [\, x, ix \,] + [\, x, iy \,] + [\, y, ix \,] + [\, y, iy \,] = [\, x, iy \,] + [\, y, ix \,]$, hence $[\, x, iy \,] = -[\, y, ix \,] = -[\, ix, y \,]$. Thus

- $\langle x+y, z \rangle = [\, x+y, z \,] + i [\, x+y, iz \,] = [\, x, z \,] + [\, y, z \,] + i ([\, x, iz \,] + [\, y, iz \,]) = ([\, x, z \,] + i [\, x, iz \,]) + ([\, y, z \,] + i [\, y, iz \,]) = \langle x, z \rangle + \langle y, z \rangle$
- $\langle cx, y \rangle = [\, (a+ib)x, y \,] + i [\, (a+ib)x, iy \,] = a [\, x, y \,] + b [\, ix, y \,] + ia [\, x, iy \,] + ib [\, ix, iy \,] = a ([\, x, y \,] + i [\, x, iy \,]) + ib ([\, ix, iy \,] - i [\, ix, y \,]) = a ([\, x, y \,] + i [\, x, iy \,]) + ib (-[\, x, -y \,] + i [\, x, iy \,]) = (a+ib) ([\, x, y \,] + i [\, x, iy \,]) = c \langle x, y \rangle$
- $\overline{\langle y, x \rangle} = [\, y, x \,] - i [\, y, ix \,] = [\, x, y \,] + i [\, iy, x \,] = \langle x, y \rangle$
- Assuming $x \ne 0$, we have $\langle x, x \rangle = [\, x, x \,] + i [\, x, ix \,] = [\, x, x \,] > 0$.

As x, y, z, c are arbitrary, $\langle \cdot, \cdot \rangle$ is a complex inner product on V.

6.2.2(g). Apply the Gram–Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for Span(S). Then normalize the vectors in this basis to obtain an orthonormal basis β for Span(S), and compute the Fourier coefficients of the given vector relative to β. Finally, use Theorem 6.5 to verify your result.

$$V = M_{2\times 2}(\mathbb{R}),\ S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}, \text{ and } A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$$

Solution: Recall that the inner product on $V = M_{2\times 2}(\mathbb{R})$ is defined as $\langle A, B \rangle = \operatorname{tr}(B^\mathsf{T} A) = \sum_{i,j} A_{ij} B_{ij}$.

(a) Applying the Gram–Schmidt process, we have

$$w_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} \quad \text{so } v_1 = w_1 \text{ with } \| v_1 \| = 6$$

$$w_2 = \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix} \quad \text{so } v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\| v_1 \|^2} v_1 = w_2 - \frac{36}{36} v_1 = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} \text{ with } \| v_2 \| = 6\sqrt{2}$$

$$w_3 = \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \quad \text{so } v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\| v_1 \|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\| v_2 \|^2} v_2 = w_3 - \frac{-72}{36} v_1 - \frac{-72}{72} v_2 = \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix} \text{ with } \| v_3 \| = 9\sqrt{2}$$

(b) By normalizing the vectors we have $\beta = \{ e_1, e_2, e_3 \}$ with

$$e_1 = v_1 / \| v_1 \| = \begin{pmatrix} 1/2 & 5/6 \\ -1/6 & 1/6 \end{pmatrix}$$
$$e_2 = v_2 / \| v_2 \| = \begin{pmatrix} -\sqrt{2}/3 & \sqrt{2}/3 \\ -\sqrt{2}/2 & -\sqrt{2}/6 \end{pmatrix}$$
$$e_3 = v_3 / \| v_3 \| = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/6 \\ \sqrt{2}/3 & -\sqrt{2}/3 \end{pmatrix}$$

(c) The Fourier coefficients of A relative to β are

$$c_1 = \langle A, e_1 \rangle = 24$$
$$c_2 = \langle A, e_2 \rangle = 6\sqrt{2}$$
$$c_3 = \langle A, e_3 \rangle = -9\sqrt{2}$$

(d) $\sum_{k=1}^{3} c_k e_k = 24 \begin{pmatrix} 1/2 & 5/6 \\ -1/6 & 1/6 \end{pmatrix} + 6\sqrt{2} \begin{pmatrix} -\sqrt{2}/3 & \sqrt{2}/3 \\ -\sqrt{2}/2 & -\sqrt{2}/6 \end{pmatrix} - 9\sqrt{2} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/6 \\ \sqrt{2}/3 & -\sqrt{2}/3 \end{pmatrix} = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix} = A$. Hence the result is verified on A.

6.2.6. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y \rangle \neq 0$.

Solution: Since W is finite-dimensional, it has a finite basis $\beta = \{ w_1, \ldots, w_n \}$ for some $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in W$. By Gram–Schmidt process we may assume that β is orthonormal.

Let $x \in V \setminus W$. Then $x \notin \operatorname{Span}(\beta)$. So $x \neq \sum_{k=1}^{n} \langle x, w_k \rangle w_k \in \operatorname{Span}(\beta)$, or $y = x - \sum_{k=1}^{n} \langle x, w_k \rangle w_k \neq 0$. Then

- For each $w_k \in \beta$, $\langle y, w_k \rangle = \langle x - \sum_{l=1}^{n} \langle x, w_l \rangle w_l, w_k \rangle = \langle x, w_k \rangle - \sum_{l=1}^{n} \langle x, w_l \rangle \langle w_l, w_k \rangle = \langle x, w_k \rangle - \langle x, w_k \rangle = 0$, so $y \in \operatorname{Span}(\beta)^\perp = W^\perp$
- $\langle x, y \rangle = \langle y + \sum_{k=1}^{n} \langle x, w_k \rangle w_k, y \rangle = \langle y, y \rangle + \sum_{k=1}^{n} \langle x, w_k \rangle \langle w_k, y \rangle = \| y \|^2 \neq 0$

As x is arbitrary, for all $x \in V \setminus W$ there exists $y \in W^\perp$ such that $\langle x, y \rangle \neq 0$.

Note

This does not hold (in general) if W is not finite-dimensional: consider $V = C([0,1])$ equipped with inner product $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$, and $W = \{ f \in V : f(0) = 0 \}$, $x \in V$ being the constant 1 function. It is easy to verify that $\langle x, y \rangle = 0$ for all $y \in W^\perp$.

6.2.10. Let W be a finite-dimensional subspace of an inner product space V. Prove that there exists a projection T on W along $W^\perp$ that satisfies $\mathsf{N}(T) = W^\perp$. In additional, prove that $\| T(x) \| \leq \| x \|$ for all $x \in V$.

Solution: Since W is finite-dimensional, it has a finite basis $\beta = \{ w_1, \ldots, w_n \}$ for some $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in W$. By Gram–Schmidt process we may assume that β is orthonormal.

Let $T : V \to V$ be defined as $T(v) = \sum_{k=1}^{n} \langle v, w_k \rangle w_k$ for all $v \in V$. By the property of inner product, T is linear. Since $w_k \in W$ for all k, we have $T(v) \in W$ for all $v \in V$.

Before showing that T is the projection on W along $W^\perp$, we first need to show that $V = W \oplus W^\perp$, so that it is meaningful to consider such projection. Note that this is exactly Question 6.2.13(d).

Trivially, $W + W^\perp \subseteq V$. Let $v \in V$. Then $Tv \in W$. Also, for each $k \in \{ 1, \ldots, n \}$, $\langle v - Tv, w_k \rangle = \langle v, w_k \rangle - \sum_{l=1}^{n} \langle v, w_l \rangle \langle w_l, w_k \rangle = \langle v, w_k \rangle - \langle v, w_k \rangle = 0$. This implies that for all $w \in W$ with $w = \sum_{k=1}^{n} c_k w_k$ for some scalars $c_1, \ldots, c_n$ we have $\langle v - Tv, w \rangle = \sum_{k=1}^{n} c_k \langle v - Tv, w_k \rangle = 0$. So $v - Tv \in W^\perp$, and thus $v = Tv + (v - Tv) \in W + W^\perp$. As v is arbitrary, $V \subseteq W + W^\perp$ and thus $V = W + W^\perp$.

Trivially $\{0\} \subseteq W \cap W^\perp$. Let $v \in W \cap W^\perp$. Then $v \in W$ and for all $w \in W$ we have $\langle v, w \rangle = 0$. In particular, $\langle v, v \rangle = 0$ and so $v = 0$. This implies that $W \cap W^\perp \subseteq \{0\}$ and thus $W \cap W^\perp = \{0\}$.

Hence $V = W \oplus W^\perp$.

We now show that T is the required map. Let $v \in V$. By the property of direct sum, there exist $w \in W$ such that $v - w \in W^\perp$. In particular, $\langle v - w, w_k \rangle = 0$ for all $w_k \in \beta$. Then $T(v) = T(w) + T(v - w) = (\sum_{k=1}^{n} \langle w, w_k \rangle w_k) + (\sum_{k=1}^{n} \langle v - w, w_k \rangle w_k) = \sum_{k=1}^{n} \langle w, w_k \rangle w_k = w$ since $w \in W$ and β is an orthonormal basis of W. By the definition, T is the projection on W along $W^\perp$.

That $\mathsf{N}(T) = W^\perp$ follows from Question 2.1.26(b) in Homework 2.

For all $x \in V$, we have $Tx \in W$ and $x - Tx \in W^\perp$, and so $\| x \|^2 = \| Tx + (x - Tx) \|^2 = \| Tx \|^2 + \| x - Tx \|^2 + 2\Re \langle x, x - Tx \rangle = \| Tx \|^2 + \| x - Tx \|^2 \geq \| Tx \|^2$, which implies that $\| Tx \| \leq \| x \|$.

Note

This does not hold (in general) if W is not finite-dimensional: with the same example in the remark of Question 6.2.6, it is easy to verify that $W + W^\perp \neq V$, and so there is no such projection.

See also Question 6.2.13, 6.2.16, 6.1.9, 6.1.10.

6.2.13. Let V be an inner product space, S and S_0 be subsets of V, and W be a finite-dimensional subspace of V. Prove the following results.

(a) $S_0 \subseteq S$ implies that $S^\perp \subseteq S_0^\perp$

(b) $S \subseteq (S^\perp)^\perp$; so $\mathrm{Span}(\ S\ ) \subseteq (S^\perp)^\perp$

(c) $W = (W^\perp)^\perp$

(d) $V = W \oplus W^\perp$

Solution:

(a) Let $v \in S^\perp$. Then for all $s \in S$, $\langle s, v \rangle = 0$. Since $S \supseteq S_0$, this implies that $\langle s, v \rangle = 0$ for all $s \in S_0$, and thus $v \in S_0^\perp$. As v is arbitrary, $S^\perp \subseteq S_0^\perp$

(b) Let $v \in S$. Then for all $u \in S^\perp$, $\langle u, v \rangle = 0$, so $v \in (S^\perp)^\perp$. As v is arbitrary, $S \subseteq (S^\perp)^\perp$.

As $(S^\perp)^\perp$ is a subspace of V, by the property of span we have $\mathrm{Span}(\ S\ ) \subseteq (S^\perp)^\perp$.

(c) By the previous part we have $W \subseteq (W^\perp)^\perp$.

Let $x \in (W^\perp)^\perp$. Suppose $x \notin W$. Then by the result of Question 6.2.6 there exists $y \in W^\perp$ with $\langle x, y \rangle \neq 0$. This contradicts with the assumption that $x \in (W^\perp)^\perp$, as it would imply that $\langle x, y \rangle = 0$. Hence $x \in W$. As x is arbitrary, $(W^\perp)^\perp \subseteq W$.

Hence $W = (W^\perp)^\perp$.

Note

The inclusion $W \supseteq (W^\perp)^\perp$ does not hold (in general) if W is not finite-dimensional: with the same example in the remark of Question 6.2.6, it is easy to verify that $(W^\perp)^\perp \supsetneq W$. You can however show that $(W^\perp)^\perp = \overline{W}$ with the induced topology.

(d) See the proof of Question 6.2.10 above.

Note

See also the remark of Question 6.2.10.

Note

See also this and this.

6.2.15. Let V be a finite-dimensional inner product space over $\mathbb{F}$

(a) *Parseval's Identity.* Let $\{ v_1, v_2, \ldots, v_n \}$ be an orthonormal basis for V. For any $x, y \in V$ prove that

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, x_i \rangle}$$

(b) Use (a) to prove that if β is an orthonormal basis for V with inner product $\langle \cdot, \cdot \rangle$, then for any $x, y \in V$

$$\langle \phi_\beta(x), \phi_\beta(y) \rangle' = \langle [x]_\beta, [y]_\beta \rangle' = \langle x, y \rangle$$

where $\langle \cdot, \cdot \rangle'$ is the standard inner product on $\mathbb{F}^n$.

Solution:

(a) Let $x, y \in V$. Since $\{ v_1, \ldots, v_n \}$ is an orthonormal basis for V, $x = \sum_{i=1}^n \langle x, v_i \rangle v_i$ and $y = \sum_{i=1}^n \langle y, v_i \rangle v_i$. So $\langle x, y \rangle = \left\langle \sum_{i=1}^n \langle x, v_i \rangle v_i, \sum_{j=1}^n \langle y, v_j \rangle v_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^n \langle x, v_i \rangle \overline{\langle y, v_j \rangle} \langle v_i, v_j \rangle = \sum_{i=1}^n \langle x, v_i \rangle \overline{\langle y, v_i \rangle}$.

(b) Let $x, y \in V$, and $\beta = \{ v_1, \ldots, v_n \}$. Then $[x]_\beta = \begin{pmatrix} c_1 & \ldots & c_n \end{pmatrix}^\mathsf{T}$ with $x = \sum_{i=1}^n c_i v_i$. Since β is orthonormal, $c_i = \langle x, v_i \rangle$, so $[x]_\beta = \begin{pmatrix} \langle x, v_1 \rangle & \ldots & \langle x, v_n \rangle \end{pmatrix}^\mathsf{T}$. Similarly, $[y]_\beta = \begin{pmatrix} \langle y, v_1 \rangle & \ldots & \langle y, v_n \rangle \end{pmatrix}^\mathsf{T}$. Hence $\langle [x]_\beta, [y]_\beta \rangle' = \sum_{i=1}^n ([x]_\beta)_i \overline{([y]_\beta)_i} = \sum_{i=1}^n \langle x, v_i \rangle \overline{\langle y, v_i \rangle} = \langle x, y \rangle$ where the last equality comes from the previous part. As x, y, β are arbitrary, $\langle [x]_\beta, [y]_\beta \rangle' = \langle x, y \rangle$ for all $x, y \in V$ and orthonormal basis β for V.

Note

Generalization to infinite-dimensional spaces exists, most remarkably on complete orthogonal systems (e.g. Fourier).

6.2.16. (a) *Bessel's Inequality.* Let V be an inner product space, and let $S = \{ v_1, v_2, \ldots, v_n \}$ be an orthonormal subset of V. Prove that for any $x \in V$ we have

$$\| x \|^2 \geq \sum_{i=1}^{n} | \langle x, v_i \rangle |^2$$

(b) In the context of (a), prove that Bessel's inequality is an equality if and only if $x \in \operatorname{Span}(S)$.

Solution:

(a) For each $x \in V$ we have

$$\begin{aligned} 0 \leq \left\| x - \sum_{i=1}^{n} \langle x, v_i \rangle v_i \right\|^2 &= \| x \|^2 + \left\| \sum_{i=1}^{n} \langle x, v_i \rangle v_i \right\|^2 - \left\langle x, \sum_{i=1}^{n} \langle x, v_i \rangle v_i \right\rangle - \left\langle \sum_{i=1}^{n} \langle x, v_i \rangle v_i, x \right\rangle \\ &= \| x \|^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, v_i \rangle \overline{\langle x, v_j \rangle} \langle v_i, v_j \rangle - \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle x, v_i \rangle} - \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle x, v_i \rangle} \\ &= \| x \|^2 + \sum_{i=1}^{n} | \langle x, v_i \rangle |^2 - 2 \sum_{i=1}^{n} | \langle x, v_i \rangle |^2 \\ &= \| x \|^2 - \sum_{i=1}^{n} | \langle x, v_i \rangle |^2 \end{aligned}$$

or equivalently $\| x \|^2 \geq \sum_{i=1}^n | \langle x, v_i \rangle |^2$.

(b) By the proof of the previous part, the equality holds if and only if $\| x - \sum_{i=1}^{n} \langle x, v_i \rangle v_i \|^2 = 0$, or equivalently $x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$. As S is orthonormal, this holds if and only if $x \in \mathrm{Span}(S)$.

Note

See also this.

6.2.17. Let T be a linear operator on an inner product space V. If $\langle T(x), y \rangle = 0$ for all $x, y \in V$, prove that $T = T_0$. In fact, prove this result if the equality holds for all x and y in some basis for V.

Solution:

(a) Let $x \in V$. By assumption on T, by choosing $y = T(x) \in V$ we have $0 = \langle T(x), T(x) \rangle = \| T(x) \|^2$, so $T(x) = 0$. As x is arbitrary, $T = 0$.

(b) Let β be a basis for V, and assume that $\langle T(x), y \rangle = 0$ for all $x, y \in \beta$.

Let $x \in \beta$. Since β is a basis of V, there exists $n \in \mathbb{N}$, distinct $\beta_1, \ldots, \beta_n \in \beta$, and $c_1, \ldots, c_n \in \mathbb{F}$ such that $T(x) = \sum_{i=1}^{n} c_i \beta_i$. Then $\| T(x) \|^2 = \langle T(x), T(x) \rangle = \sum_{i=1}^{n} \overline{c_i} \langle T(x), \beta_i \rangle = 0$. This implies that $T(x) = 0$. As $x \in \beta$ is arbitrary, T vanishes on a basis and so is the zero map.

Practice Problems

6.1.1. Label the following statements as true or false.

(a) An inner product is a scalar-valued function on the set of ordered pairs of vectors.

(b) An inner product space must be over the field of real or complex numbers.

(c) An inner product is linear in both components.

(d) There is exactly one inner product on the vector space $\mathbb{R}^n$.

(e) The triangle inequality only holds in finite-dimensional inner product spaces.

(f) Only square matrices have a conjugate-transpose.

(g) If x, y, and z are vectors in an inner product space such that $\langle x, y \rangle = \langle x, z \rangle$, then $y = z$.

(h) If $\langle x, y \rangle = 0$ for all x in an inner product space, then $y = 0$.

Solution:

(a) True

(b) False. In the definition of inner product space, we only need the field to be ordered (so that $\langle x, x \rangle > 0$ is well-defined) and have a conjugation defined (which may be the identity map, as in the real case). The restriction on real and complex numbers is mostly to make things easier. See this wiki article and this answer on MSE.

(c) False. A complex inner product is not linear in the second component (it is conjugate linear)

(d) False

(e) False

(f) False

(g) False

(h) True

6.1.3. In $C([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g \rangle$, $\| f \|$, $\| g \|$, and $\| f + g \|$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t$. Then verify both the Cauchy–Schwarz inequality and the triangle inequality.

Solution:

- $\langle f, g \rangle = \int_0^1 t \, \mathrm{d}e^t = te^t|_0^1 - \int_0^1 e^t \, \mathrm{d}t = e - (e-1) = 1$
- $\| f \|^2 = \int_0^1 t^2 \, \mathrm{d}t = \frac{1}{3}$, so $\| f \| = \sqrt{\frac{1}{3}} \approx 0.577$
- $\| g \|^2 = \int_0^1 (e^t)^2 \, \mathrm{d}t = \frac{1}{2}(e^2 - 1)$, so $\| g \| = \sqrt{\frac{1}{2}(e^2-1)} \approx 1.787$
- $\| f + g \|^2 = \int_0^1 (t + e^t)^2 \, \mathrm{d}t = \int_0^1 t^2 \, \mathrm{d}t + \int_0^1 e^{2t} \, \mathrm{d}t + 2\int_0^1 te^t \, \mathrm{d}t = \frac{1}{3} + \frac{1}{2}(e^2-1) + 2 = \frac{1}{3} + \frac{1}{2}(e^2+3)$, so $\| f + g \| = \sqrt{\frac{1}{3} + \frac{1}{2}(e^2+3)} \approx 2.351$

Then $\| f \| \| g \| > 0.57 \times 1.78 = 1.0146 > 1 = | \langle f, g \rangle |$ and $\| f \| + \| g \| > 0.575 + 1.785 = 2.36 > \| f + g \|$. So Cauchy–Schwarz inequality and triangular inequality are verified on the pair f, g.

Note

There are better ways to compare the square roots than the numeric one presented here.

6.1.9. Let β be a basis for a finite-dimensional inner product space.

(a) Prove that if $\langle x, z \rangle = 0$ for all $z \in \beta$, then $x = 0$

(b) Prove that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $x = y$.

Solution: Since β is a basis of a finite-dimensional space, it is also finite. Hence we may assume that $\beta = \{ v_1, \ldots, v_n \}$ for some $n \in \mathbb{N}$.

(a) As $x \in V$, there exist $c_1, \ldots, c_n \in \mathbb{F}$ such that $x = \sum_{i=1}^n c_i v_i$. Then $\langle x, x \rangle = \sum_{i=1}^n \overline{c_i} \langle x, v_i \rangle = 0$ as $v_1, \ldots, v_n \in \beta$. This implies that $x = 0$.

(b) By the linearity of inner product we have $\langle x - y, z \rangle = \langle x, z \rangle - \langle y, z \rangle = 0$ for all $z \in \beta$. By the previous part, $x - y = 0$, and so $x = y$.

6.1.10. Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\| x+y \|^2 = \| x \|^2 + \| y \|^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.

Solution: Since x, y are orthogonal, $\langle x, y \rangle = 0$. So by the result of Question 6.1.19(a), $\| x + y \|^2 = \| x \|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \| y \|^2 = \| x \|^2 + \| y \|^2$.

As two vectors in $\mathbb{R}^2$ are orthogonal if and only if they are perpendicular to each other, and the norm induced by the standard inner product on $\mathbb{R}^2$ is exactly the length of the vector, we have the following result:

> The square of the length of the hypotenuse in a right triangular is the sum of the squares of the lengths of its sides

which is exactly the Pythagorean theorem.

6.1.11. Prove the parallelogram law on an inner product space V; that is, show that

$$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2 \quad \text{for all } x, y \in V$$

What does this equation state about parallelograms in $\mathbb{R}^2$?

Solution: By the result of Question 6.1.19(a), we have $\| x + y \|^2 + \| x - y \|^2 = \Big(\| x \|^2 + \langle x, y \rangle + \langle y, x \rangle + \| y \|^2 \Big) + \Big(\| x \|^2 - \langle x, y \rangle - \langle y, x \rangle + \| y \|^2 \Big) = 2 \| x \|^2 + 2 \| y \|^2$ for all $x, y \in V$.

Consider a parallelogram with one of its angle at the origin, and $x, y \in \mathbb{R}^2$ be the two sides of this angle. Then $x + y, x - y$ are vectors of the two diagonals of the parallelogram. This equation implies that the sum of squares of the lengths of the diagonals in a parallelogram is exactly the sum of squares of the lengths of its sides.

6.1.20. Let V be an inner product space over $\mathbb{F}$. Prove the *polar identities*: For all $x, y \in V$,

(a) $\langle x, y \rangle = \frac{1}{4} \| x+y \|^2 - \frac{1}{4} \| x-y \|^2$ if $\mathbb{F} = \mathbb{R}$;

(b) $\langle x, y \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k \left\| x + i^k y \right\|^2$ if $\mathbb{F} = \mathbb{C}$, where $i^2 = -1$

Solution:

(a) $\frac{1}{4} \| x+y \|^2 - \frac{1}{4} \| x-y \|^2 = \frac{1}{4} \Big(\| x \|^2 + \langle x, y \rangle + \langle y, x \rangle + \| y \|^2 \Big) - \frac{1}{4} \Big(\| x \|^2 - \langle x, y \rangle - \langle y, x \rangle + \| y \|^2 \Big) = \langle x, y \rangle$

(b)

$$\begin{aligned}
\frac{1}{4} \sum_{k=1}^{4} i^k \left\| x + i^k y \right\|^2 = \frac{1}{4} \Big[& i \cdot \Big(\| x \|^2 + \| y \|^2 + \langle x, iy \rangle + \langle iy, x \rangle \Big) \\
& - 1 \cdot \Big(\| x \|^2 + \| y \|^2 + \langle x, -y \rangle + \langle -y, x \rangle \Big) \\
& - i \cdot \Big(\| x \|^2 + \| y \|^2 + \langle x, -iy \rangle + \langle -iy, x \rangle \Big) \\
& + 1 \cdot \Big(\| x \|^2 + \| y \|^2 + \langle x, y \rangle + \langle y, x \rangle \Big) \Big] \\
= \frac{1}{4} & [(\langle x, y \rangle - \langle y, x \rangle) + (\langle x, y \rangle + \langle y, x \rangle) + (\langle x, y \rangle - \langle y, x \rangle) + (\langle x, y \rangle + \langle y, x \rangle)] \\
= \langle x, & y \rangle
\end{aligned}$$

Note

You can show the following theorem often attributed to von Neumann: if a norm on a real or complex vector space satisfies parallelogram law (Question 6.1.11), then it is induced by the inner product constructed by the polar identity.

You can also show the following: if $\omega \in \mathbb{C} \backslash \{\pm 1\}$ satisfies $\omega^n = 1$ for some integer $n \geq 3$, then $\langle x, y \rangle = \frac{1}{n} \sum_{k=1}^{n} \omega^k \left\| x + \omega^k y \right\|^2$ for all x, y in a complex inner product space.

See also this.

6.1.21. Let A be an $n \times n$ matrix. Define

$$A_1 = \frac{1}{2}(A + A^*) \quad \text{and} \quad A_2 = \frac{1}{2i}(A - A^*)$$

(a) Prove that $A_1^* = A_1$, $A_2^* = A_2$, and $A = A_1 + iA_2$. Would it be reasonable to define A_1 and A_2 to be the real and imaginary parts, respectively, of the matrix A?

(b) Let A be an $n \times n$ matrix. Prove that the representation in (a) is unique. That is, prove that if $A = B_1 + iB_2$, where $B_1^* = B_1$ and $B_2^* = B_2$, then $B_1 = A_1$ and $B_2 = A_2$.

Solution:

(a)
- $A_1^* = \left(\frac{1}{2}(A + A^*)\right)^* = \frac{1}{2}(A^* + (A^*)^*) = \frac{1}{2}(A^* + A) = A_1$
- $A_2^* = \left(\frac{1}{2i}(A - A^*)\right)^* = \frac{1}{-2i}(A^* - (A^*)^*) = \frac{1}{2i}(A - A^*) = A_2$
- $A_1 + iA_2 = \frac{1}{2}(A + A^*) + \frac{i}{2i}(A - A^*) = \frac{1}{2}(A + A) = A$

Note that for a complex number $z \in \mathbb{C}$ we have $\Re(z) = \frac{1}{2}(z + z^*)$ and $\Im(z) = \frac{1}{2i}(z - z^*)$ with $z^* = \overline{z}$. Hence this definition extends the usual definition of real part and imaginary part to matrices, and thus is a reasonable definition.

(b) Suppose $A = B_1 + iB_2$ with $B_1 = B_1^*$ and $B_2 = B_2^*$. Then $A^* = (B_1 + iB_2)^* = B_1^* - iB_2^* = B_1 - iB_2$. So $A_1 = \frac{1}{2}(A + A^*) = \frac{1}{2}\left((B_1 + iB_2) + (B_1 - iB_2) \right) = B_1$, $A_2 = \frac{1}{2i}(A - A^*) = \frac{1}{2i}\left((B_1 + iB_2) - (B_1 - iB_2) \right) = B_2$.

Note

See also this and perhaps more popularly this.

6.1.28. Let V be a complex inner product space with an inner product $\langle \cdot, \cdot \rangle$. Let $[\, \cdot, \cdot \,]$ be the real-valued function such that $[\, x, y \,]$ is the real part of the complex number $\langle x, y \rangle$ for all $x, y \in V$. Prove that $[\, \cdot, \cdot \,]$ is an inner product for V, where V is regarded as a vector space over $\mathbb{R}$. Prove, furthermore, that $[\, x, ix \,] = 0$ for all $x \in V$.

Solution: Denote the real vector space as $V_{\mathbb{R}}$. For all $x, y, z \in V_{\mathbb{R}}$ and $c \in \mathbb{R}$,

- $[\, x+y,\, z\,] = \Re\langle\, x+y,\, z\,\rangle = \Re(\langle\, x,\, z\,\rangle + \langle\, y,\, z\,\rangle) = \Re\langle\, x,\, z\,\rangle + \Re\langle\, y,\, z\,\rangle = [\, x,\, z\,] + [\, y,\, z\,]$
- $[\, cx,\, y\,] = \Re\langle\, cx,\, y\,\rangle = \Re(c\langle\, x,\, y\,\rangle) = c\Re\langle\, x,\, y\,\rangle = c[\, x,\, y\,]$ as $c \in \mathbb{R}$
- $[\, y,\, x\,] = \Re\langle\, y,\, x\,\rangle = \frac{1}{2}\left(\langle\, y,\, x\,\rangle + \overline{\langle\, y,\, x\,\rangle}\right) = \frac{1}{2}(\langle\, x,\, y\,\rangle + \langle\, y,\, x\,\rangle) = \Re\langle\, x,\, y\,\rangle = [\, x,\, y\,]$
- Assuming $x \neq 0$, we have $[\, x,\, x\,] = \Re\langle\, x,\, x\,\rangle = \langle\, x,\, x\,\rangle > 0$ as $\langle\, x,\, x\,\rangle \in \mathbb{R}$.

As x, y, z, c are arbitrary, $[\,\cdot,\,\cdot\,]$ is a real inner product on $V_{\mathbb{R}}$.
For all $x \in V_{\mathbb{R}}$, we also have $[\, x,\, ix\,] = \Re\langle\, x,\, ix\,\rangle = \frac{1}{2}\left(\langle\, x,\, ix\,\rangle + \overline{\langle\, x,\, ix\,\rangle}\right) = \frac{1}{2}(-i\langle\, x,\, x\,\rangle + i\langle\, x,\, x\,\rangle) = 0$.

Note

See also Question 6.1.29.

6.2.1. Label the following statements as true or false.

(a) The Gram–Schmidt orthogonalization process allows us to construct an orthonormal set from an arbitrary set of vectors.

(b) Every nonzero finite-dimensional inner product space has an orthonormal basis.

(c) The orthogonal complement of any set is a subspace.

(d) If $\{\, v_1, v_2, \ldots, v_n\,\}$ is a basis for an inner product space V, then for any $x \in V$ the scalars $\langle\, x,\, v_i\,\rangle$ are the Fourier coefficients of x.

(e) An orthonormal basis must be an ordered basis.

(f) Every orthogonal set is linearly independent.

(g) Every orthonormal set is linearly independent.

Solution:

(a) True, but note that the empty set is also by definition orthonormal. If instead of just "an orthonormal set" we want "an orthonormal set that has the same span", then the statement is false: Gram–Schmidt process is defined by induction on natural numbers and so can only gives countably many vectors. For inner product spaces with uncountable bases it may be necessary to consider induction on uncountable sets. See also this.

(b) True

(c) True

(d) True

(e) True

(f) False

(g) True

6.2.4. Let $S = \{\, (1, 0, i), (1, 2, 1)\,\}$ in $\mathbb{C}^3$. Compute $S^{\perp}$.

Solution: For $(a, b, c) \in \mathbb{C}^3$, $(a, b, c) \in S^{\perp}$ if and only if $\langle\, (a, b, c),\, (1, 0, i)\,\rangle = a - ic = 0$ and $\langle\, (a, b, c),\, (1, 2, 1)\,\rangle = a + 2b + c = 0$. This implies that $S^{\perp} = \mathsf{N}\left(\begin{pmatrix} 1 & 0 & -i \\ 1 & 2 & 1 \end{pmatrix}\right) = \mathrm{Span}(\ \{\, (2i, -1-i, 2)\,\}\ )$

6.2.14. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$ and $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$.

Solution:

(a) Let $x \in (W_1 + W_2)^{\perp}$. Then for all $w_1 \in W_1$, $w_2 \in W_2$ we have $\langle\, x,\, w_1 + w_2 \,\rangle = 0$. As W_2 is a subspace, $0 \in W_2$. So for all $w \in W_1$, we have $\langle\, x,\, w_1 \,\rangle = \langle\, x,\, w_1 + 0 \,\rangle = 0$. This implies that $x \in W_1^{\perp}$. Similarly we have $x \in W_2^{\perp}$, so $x \in W_1^{\perp} \cap W_2^{\perp}$. As x is arbitrary, $(W_1 + W_2)^{\perp} \subseteq W_1^{\perp} \cap W_2^{\perp}$.

Let $x \in W_1^{\perp} \cap W_2^{\perp}$. Then $x \in W_1^{\perp}$ and $x \in W_2^{\perp}$, so for all $w_1 \in W_1$ and $w_2 \in W_2$ we have $\langle\, x,\, w_1 \,\rangle = \langle\, x,\, w_2 \,\rangle = 0$, and so $\langle\, x,\, w_1 + w_2 \,\rangle = 0$. This implies that $x \in (W_1 + W_2)^{\perp}$. As x is arbitrary, $W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}$.

Hence $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

(b) Since the space is finite-dimensional, by the result of Question 6.2.13(c), $(W_1^{\perp} + W_2^{\perp})^{\perp} = (W_1^{\perp})^{\perp} \cap (W_2^{\perp})^{\perp} = W_1 + W_2$, which gives $(W_1 + W_2)^{\perp} = ((W_1^{\perp} + W_2^{\perp})^{\perp})^{\perp} = W_1^{\perp} + W_2^{\perp}$.

Note

Unlike the previous part, this does not hold if the space is not finite-dimensional. See Question 6.2.13(c).

6.2.18. Let $V = C([-1,1])$. Suppose that W_e and W_0 denote the subspaces of V consisting of the even and odd functions, respectively. Prove that $W_e^{\perp} = W_o$, where the inner product on V is defined by

$$\langle\, f,\, g \,\rangle = \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t$$

Solution: Let $f \in W_o$. Then for all $g \in W_e$, we have $\langle\, f,\, g \,\rangle = \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t = \int_{-1}^{0} f(t)g(t)\,\mathrm{d}t + \int_{0}^{1} f(t)g(t)\,\mathrm{d}t = -\int_{1}^{0} f(-t)g(-t)\,\mathrm{d}t + \int_{0}^{1} f(t)g(t)\,\mathrm{d}t = -\int_{0}^{1} f(t)g(t)\,\mathrm{d}t + \int_{0}^{1} f(t)g(t)\,\mathrm{d}t = 0$, so $f \in W_e^{\perp}$. As f is arbitrary, this implies that $W_o \subseteq W_e^{\perp}$.

It is easy to see that $V = W_e \oplus W_o$. As $V = W_e + W_o \subseteq W_e + W_e^{\perp} \subseteq V$, $W_e + W_e^{\perp} = V$. Also, $f \in W_e \cap W_e^{\perp}$ if and only if $\langle\, f,\, f \,\rangle = 0$ or equivalently $f = 0$, which implies that $V = W_e \oplus W_e^{\perp}$.

Let $f \in W_e^{\perp}$. Since $V = W_e \oplus W_o$, there exist $f_e \in W_e$ and $f_o \in W_o \subseteq W_e^{\perp}$ such that $f = f_e + f_o$, so $f_e + (f_o - f) = 0$ is a decomposition of 0 according to the direct sum $V = W_e \oplus W_e^{\perp}$ with $f_e \in W_e$ and $f_o - f \in W_e^{\perp}$. This implies that $f_e = 0$ and $f = f_o \in W_o$. As f is arbitrary, $W_e^{\perp} \subseteq W_o$.

Therefore $W_e^{\perp} = W_o$.

Note

Compare this with the remark in Question 6.2.13(d).