MATH2040 Homework 4
Reference Solution

5.1.2(e). For the following linear operator T on a vector space V and ordered bases β, compute $[T]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

$$\begin{aligned} V &= \mathsf{P}_3(\mathbb{R}) \\ T(a+bx+cx^2+dx^3) &= -d+(-c+d)x+(a+b-2c)x^2+(-b+c-2d)x^3 \\ \beta &= \{\ 1-x+x^3, 1+x^2, 1, x+x^2\ \} \end{aligned}$$

Solution:

$$\begin{aligned} T(1-x+x^3) &= -1+x-x^3 = -1\cdot(1-x+x^3) \\ T(1+x^2) &= -x-x^2+x^3 = 1\cdot(1-x+x^3)-1\cdot(1+x^2) \\ T(1) &= x^2 = 1\cdot(1+x^2)-1\cdot 1 \\ T(x+x^2) &= -x-x^2 = -1\cdot(x+x^2) \end{aligned}$$

so we have $[T]_\beta = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Since $1+x^2, 1 \in \beta$ are not eigenvectors of T, β is not a basis consisting of eigenvectors of T.

5.1.2(f). For the following linear operator T on a vector space V and ordered bases β, compute $[T]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

$$\begin{aligned} V &= M_{2\times 2}(\mathbb{R}) \\ T\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{pmatrix} \\ \beta &= \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\} \end{aligned}$$

Solution:

$$\begin{aligned} T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix} = -3\cdot\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ T\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = 1\cdot\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \\ T\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \\ T\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = 1\cdot\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

so we have $[T]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Since every vector in β is an eigenvector of T, β is a basis consisting of eigenvectors of T.

5.1.3(d). For the following matrix $A \in M_{n\times n}(\mathbb{F})$,

(a) Determine all the eigenvalues of A.

(b) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(c) If possible, find a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

(d) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \text{ for } \mathbb{F} = \mathbb{R}.$$

Solution:

(a) The characteristic polynomial of A is $p(t) = \det(A - tI) = -t^3 + 2t^2 - t = -t(t-1)^2$, so the eigenvalues of A are $0, 1$.

(b)
- For eigenvalue $\lambda = 0$, we have $A - \lambda I = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$, which has null space $\mathsf{N}(\ A\ ) = \operatorname{Span}\left(\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}\right)$.
- For eigenvalue $\lambda = 1$, we have $A - \lambda I = \begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix}$, which has null space $\mathsf{N}(\ A - I\ ) = \operatorname{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}\right)$.

(c) Consider $\beta = \left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Note that vectors in β are all eigenvectors of A. It is easy to see that $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is linearly independent. So β is a union of linearly independent subsets of eigenvectors corresponding to distinct eigenvalues, β is linearly independent. As $|\ \beta\ | = 3 = \dim \mathbb{F}^n$, β is a basis of $\mathbb{F}^n$.

(d) Let α be the standard basis of $\mathbb{F}^n = \mathbb{R}^3$. Then $Q = [\mathrm{Id}]^\alpha_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ is invertible. Furthermore, by definition of eigenvector we have $D = [T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is diagonal, and $D = [T]_\beta = \left([\mathrm{Id}]^\alpha_\beta\right)^{-1} [T]_\alpha [\mathrm{Id}]^\alpha_\beta = Q^{-1}AQ$.

Note

The method used to solve for the null space of a matrix is already covered in MATH1030. The steps here are also the typical approach on diagonalizing a matrix.

5.1.4(e). For the linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

$$V = \mathsf{P}_2(\mathbb{R}) \text{ and } T(f) = xf'(x) + f(2)x + f(3)$$

Solution: Let α be the standard basis of $V = \mathsf{P}_2(\mathbb{R})$. Then

$$\begin{aligned} T(1) &= 1 + x \\ T(x) &= 3 + 3x \\ T(x^2) &= 9 + 4x + 2x^2 \end{aligned}$$

so we have $[T]_\alpha = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$.

The characteristic polynomial of $[T]_\alpha$ is $p(t) = \det([T]_\alpha - tI) = -t(t-2)(t-4)$, so the eigenvalues of T are $0, 2, 4$.

- For eigenvalue $\lambda = 0$, the null space of $[T]_\alpha = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$ is $\operatorname{Span}\left(\left\{ \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \right\}\right)$, so the null space of T is $\operatorname{Span}(\ \{\ 3 - x\ \}\ )$.

- For eigenvalue $\lambda = 2$, the null space of $[T]_\alpha - 2I = \begin{pmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$ is $\text{Span}\left(\left\{ \begin{pmatrix} 3 \\ 13 \\ -4 \end{pmatrix} \right\} \right)$, so the null space of $T - 2\text{Id}$ is $\text{Span}(\ \{\ 3 + 13x - 4x^2\ \}\ )$.

- For eigenvalue $\lambda = 4$, the null space of $[T]_\alpha - 4I = \begin{pmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{pmatrix}$ is $\text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \right)$, so the null space of $T - 4\text{Id}$ is $\text{Span}(\ \{\ 1 + x\ \}\ )$.

Let $\beta = \{\ 3 - x, 3 + 13x - 4x^2, 1 + x\ \}$. As the vectors in β are eigenvectors of T with distinct eigenvalues, β is linearly independent. As $|\ \beta\ | = 3 = \dim(V)$, β is a basis of V.

By definition of eigenvectors, we have $[T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, which is diagonal.

Note

Remember to convert the matrix representations back to the vectors after solving for the null spaces.

5.1.10. Let V be a finite-dimensional vector space, and let λ be any scalar.

(a) For any ordered basis β for V, prove that $[\lambda\text{Id}_V]_\beta = \lambda I$.

(b) Compute the characteristic polynomial of λId_V.

(c) Show that λId_V is diagonalizable and has only one eigenvalue.

Solution:

(a) Since V is finite dimensional, we may assume that $\beta = \{\ v_1, \ldots, v_n\ \}$ for some $n \in \mathbb{N}$.

Then for each $k \in \{\ 1, \ldots, n\ \}$, we have $(\lambda\text{Id}_V)(v_k) = \lambda v_k$. This implies that $([\lambda\text{Id}_V]_\beta)_{ij} = \begin{cases} \lambda & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ for all $i, j \in \{\ 1, \ldots, n\ \}$ and so $[\lambda\text{Id}_V]_\beta = \lambda I$.

(b) Let β be an ordered basis of V. By the previous part, the characteristic polynomial of λId_V is $p(t) = \det([\lambda\text{Id}_V]_\beta - tI) = \det(\lambda I - tI) = \det((\lambda - t)I) = (\lambda - t)^n$

(c) By the previous part, the characteristic polynomial of λId_V has only one root λ, so λId_V has only one eigenvalue λ.

Let β be an ordered basis of V. Then by part (a), $[\lambda\text{Id}_V]_\beta = \lambda I$, which is diagonal. So λId_V is diagonalizable.

5.1.17. Let T be the linear operator on $M_{n\times n}(\mathbb{R})$ defined by $T(A) = A^\mathsf{T}$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find an ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

(d) Find an ordered basis β for $M_{n\times n}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix for $n > 2$.

Solution: We will assume that $n \geq 2$.

(a) Let $\lambda \in \mathbb{R}$ be an eigenvalue of T. Then there exists a nonzero matrix $A \in M_{n\times n}(\mathbb{R})$ such that $A^\mathsf{T} = T(A) = \lambda A$. In particular, this implies that for all $i, j \in \{\ 1, \ldots, n\ \}$ we have $A_{ji} = (A^\mathsf{T})_{ij} = (\lambda A)_{ij} = \lambda A_{ij}$ and so $A_{ij} = \lambda A_{ji} = \lambda^2 A_{ij}$, $A_{ij}(1 - \lambda^2) = 0$. Since A is nonzero, there exists $i_0, j_0 \in \{\ 1, \ldots, n\ \}$ such that $A_{i_0,j_0} \neq 0$, $A_{i_0,j_0}(1 - \lambda^2) = 0$. Thus $1 - \lambda^2 = 0$, and so $\lambda = 1$ or $\lambda = -1$.

We now show that ± 1 are indeed eigenvalues of T by noting that there exist corresponding eigenvectors:

- For $\lambda = 1$, consider $I \in M_{n\times n}(\mathbb{R})$, the identity matrix, which is nonzero. Then $T(I) = I^\mathsf{T} = I = \lambda I$.
- For $\lambda = -1$, consider the matrix $A = E_{12} - E_{21} \in M_{n\times n}(\mathbb{R})$ where E_{ij} is the matrix where the (i, j)-entry is 1 and all other entries are zero (as defined in Homework 2 Question 1.6.15). Then A is nonzero, and $T(A) = (E_{12} - E_{21})^\mathsf{T} = E_{21} - E_{12} = \lambda(E_{12} - E_{21})$

As for each of these values there is a nonzero vector that witnesses the eigenvalue relation, ± 1 are the only eigenvalues of T.

(b)
- For $A \in M_{n\times n}(\mathbb{R})$, $A \in E_1$ if and only if $A^\mathsf{T} = T(A) = 1 \cdot A = A$. So the eigenvectors corresponding to $\lambda = 1$ are the nonzero symmetric matrices in $M_{n\times n}(\mathbb{R})$.
- For $A \in M_{n\times n}(\mathbb{R})$, $A \in E_{-1}$ if and only if $A^\mathsf{T} = T(A) = -1\cdot A = -A$. So the eigenvectors corresponding to $\lambda = -1$ are the nonzero skew-symmetric matrices in $M_{n\times n}(\mathbb{R})$.

(c) Let $\beta_1 = \{ E_{11}, E_{22}, E_{12} + E_{21} \}$, $\beta_{-1} = \{ E_{12} - E_{21} \}$. It is easy to see that $\beta_1 \subseteq E_1$, $\beta_{-1} \subseteq E_{-1}$, and β_1, β_2 are linearly independent subsets in distinct eigenspaces. This implies that $\beta = \beta_1 \cup \beta_2 = \{ E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21} \}$ is linearly independent. As $|\,\beta\,| = 4 = \dim(M_{2\times 2}(\mathbb{R}))$, β is an ordered basis. Since β consists of eigenvectors of T, $[T]_\beta$ is diagonal.

(d) Let $\beta_1 = \{ E_{ij} + E_{ji} : i,j \in \{ 1,\dots,n \}, i < j \} \cup \{ E_{ii} : i \in \{ 1,\dots,n \} \} \subseteq E_1$ and $\beta_{-1} = \{ E_{ij} - E_{ji} : i,j \in \{ 1,\dots,n \}, i < j \} \subseteq E_{-1}$. It is easy to verify that β_1 and β_{-1} are linearly independent. Since β_1, β_{-1} are linearly independent subsets in distinct eigenspaces, $\beta = \beta_1 \cup \beta_{-1}$ is linearly independent. Also, $|\,\beta\,| = |\,\beta_1\,| + |\,\beta_{-1}\,| = \left(\frac{n(n-1)}{2} + n \right) + \frac{n(n-1)}{2} = n^2 = \dim(M_{n\times n}(\mathbb{R}))$, β is an ordered basis of $M_{n\times n}(\mathbb{R})$. Since β consists of eigenvectors of T, $[T]_\beta$ is diagonal.

Note

For $n = 1$, the whole proof still works except -1 is no longer an eigenvalue of T, as T degenerates to the identity operator on $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}$.

We also omit checking that β_1 and β_{-1} are linearly independent as they are self-evident.

5.1.18. Let $A, B \in M_{n\times n}(\mathbb{C})$.

(a) Prove that if B is invertible, then there exists a scalar $c \in \mathbb{C}$ such that $A + cB$ is not invertible.

(b) Find nonzero 2×2 matrices A and B such that both A and $A + cB$ are invertible for all $c \in \mathbb{C}$.

Idea:

(a) One way to determine if a matrix is invertible is to check its determinant. So for this question, we want to guarantee the existence of a complex root c for the equation $\det(A + cB) = 0$. Note that the structure of this equation is similar to that of characteristic polynomial, which we have some properties on.

(b) In order to have the desired property, we must have B not being invertible, for otherwise the conclusion of part (a) applies. It then remains to trial-and-error.

Solution:

(a) Let $z \in \mathbb{C}$. As B is invertible, B^{-1} exists, and we have $A + zB = B(B^{-1}AB + zI)B^{-1}$. Furthermore, by the property of determinant, $\det(A + zB) = \det(B)\det(B^{-1}AB + zI)\det(B^{-1}) = \det(B^{-1}AB + zI) = p(-z)$, with $p(t) = \det(B^{-1}AB - tI)$ being the characteristic polynomial of $B^{-1}AB$. Since $B^{-1}AB \in M_{n\times n}(\mathbb{C})$, $p(t)$ is a polynomial of degree $n \geq 1$, so by the fundamental theorem of algebra p has a complex root.

Let $c \in \mathbb{C}$ be such that $-c$ is a root of p. Then $0 = p(-c) = \det(A + cB)$. By the property of determinant, $A + cB$ is not invertible.

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A, B \in M_{2\times 2}(\mathbb{C})$ and are nonzero. It is easy to see that A is invertible, and for all $c \in \mathbb{C}$ we have $A + cB = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, which has determinant $\det(A + cB) = \det \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = 1 \neq 0$. This implies that $A + cB$ is invertible for all $c \in \mathbb{C}$.

5.1.20. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

Solution: By definition, $f(t) = \det(A - tI)$. By the assumption, we have $f(0) = a_0$. So $a_0 = f(0) = \det(A - 0 \cdot I) = \det(A)$. By the property of determinant, A is invertble if and only if $\det(A) \neq 0$, which holds if and only if $a_0 \neq 0$.

5.2.3(a). For the following linear operator T on a vector space V, test T for dagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

$$V = \mathsf{P}_3(\mathbb{R}) \text{ and } T \text{ is defined by } T(f) = f'(x) + f''(x)$$

Solution: Let $\alpha = \{ 1, x, x^2, x^3 \}$ be the standard basis. Then

$$\begin{aligned} T(1) &= 0 \\ T(x) &= 1 \\ T(x^2) &= 2 + 2x \\ T(x^3) &= 6x + 3x^2 \end{aligned}$$

so $[T]_\alpha = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The characteristic polynomial of T is then $p(t) = \det([T]_\alpha - tI) = t^4$, so the only eigenvalue of T is 0, which has algebraic multiplicity $m_0 = 4$.

The null space of $[T]_\alpha$ is $\mathsf{N}\left([T]_\alpha \right) = \operatorname{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \right)$. This means that the geometric multiplicity of the eigenvalue $\lambda = 0$ is $\gamma_0 = 1$. As T has only one eigenvalue and $m_0 \neq \gamma_0$, T is not diagonalizable.

Note

You can also note that T is not the identity map on V but has only one eigenvalue, and that T does not map nonconstant polynomial to zero by considering their degrees. See also Question 5.1.11.

5.2.3(d). For the following linear operator T on a vector space V, test T for dagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

$$V = \mathsf{P}_2(\mathbb{R}) \text{ and } T \text{ is defined by } T(f) = f(0) + f(1)(x + x^2)$$

Solution: Let α be the standard basis of $V = \mathsf{P}_2(\mathbb{R})$. Then

$$\begin{aligned} T(1) &= 1 + x + x^2 \\ T(x) &= x + x^2 \\ T(x^2) &= x + x^2 \end{aligned}$$

so $[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

The characteristic polynomial of T is then $p(t) = \det([T]_\alpha - tI) = -t^3 + 3t^2 - 2t = -t(t-1)(t-2)$, so T has 3 distinct eigenvalues $0, 1, 2$. Since $\dim(V) = 3$, T is diagonalizable.

The eigenspaces of $[T]_\alpha$ and of T on these eigenvalues are, respectively,

- $E_0' = \mathsf{N}\left([T]_\alpha \right) = \operatorname{Span}\left(\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\} \right)$, so $E_0 = \operatorname{Span}(\{ x - x^2 \})$

- $E_1' = \mathsf{N}(\ [T]_\alpha - I\ ) = \operatorname{Span}\left(\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\} \right)$, so $E_1 = \operatorname{Span}(\ \{\ 1 - x - x^2\ \}\ )$
- $E_2' = \mathsf{N}(\ [T]_\alpha - 2I\ ) = \operatorname{Span}\left(\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \right)$, so $E_2 = \operatorname{Span}(\ \{\ x + x^2\ \}\ )$

Let $\beta = \{\ x - x^2, 1 - x - x^2, x + x^2\ \}$. Since the vectors in β are eigenvectors corresponding to distinct eigenvalues, β is linearly independent. As $|\ \beta\ | = 3 = \dim(V)$, β is a basis of V. Hence β is a basis such that $[T]_\beta$ is diagonal.

Note

$[T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

5.2.3(e). For the following linear operator T on a vector space V, test T for dagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

$$V = \mathbb{C}^2 \text{ and } T \text{ is defined by } T(z, w) = (z + iw, iz + w)$$

Solution: Let $\alpha = \{\ (1,0), (0,1)\ \}$ be the standard basis of $V = \mathbb{C}^2$. Then

$$\begin{aligned} T(1,0) &= (1, i) = 1 \cdot (1,0) + i \cdot (0,1) \\ T(0,1) &= (i, 1) = i \cdot (1,0) + 1 \cdot (0,1) \end{aligned}$$

so $[T]_\alpha = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.

The characteristic polynomial of T is then $p(t) = \det([T]_\alpha - tI) = t^2 - 2t + 2 = (t - 1 - i)(t - 1 + i)$, so T has 2 distinct eigenvalues $1 - i, 1 + i$. As $\dim(V) = 2$, T is diagonalizable.

The eigenspaces of $[T]_\alpha$ and of T on these eigenvalues are, respectively,

- $E_{1-i}' = \mathsf{N}(\ [T]_\alpha - (1-i)I\ ) = \operatorname{Span}\left(\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \right)$, so $E_{1-i} = \operatorname{Span}(\ \{\ (1,-1)\ \}\ )$
- $E_{1+i}' = \mathsf{N}(\ [T]_\alpha - (1+i)I\ ) = \operatorname{Span}\left(\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \right)$, so $E_{1+i} = \operatorname{Span}(\ \{\ (1,1)\ \}\ )$

Let $\beta = \{\ (1,-1), (1,1)\ \}$. Since the vectors in β are eigenvectors corresponding to distinct eigenvalues, β is linearly independent. As $|\ \beta\ | = 2 = \dim(V)$, β is a basis of V. Hence β is a basis such that $[T]_\beta$ is diagonal.

Note

$[T]_\beta = \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}$.

5.2.7. For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, find an expression for A^n, where n is an arbitrary positive integer.

Idea: Although we can compute a few powers manually and try to find the pattern, it would be easier if we can simplify the computation. In particular, if A is diagonal, we have by simple calculation that A^n is also diagonal with its entries raised to the same power. The same process still works if A is diagonalizable.

Solution: The characteristic polynomial of A is $p(t) = \det(A - tI) = t^2 - 4t - 5 = (t+1)(t-5)$. So A has 2 eigenvalues $-1, 5$. As $\dim(\mathbb{R}^2) = 2$, A is diagonalizable.

- For eigenvalue $\lambda = -1$, the null space of $A - \lambda I = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$ is $\operatorname{Span}\left(\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \right)$.
- For eigenvalue $\lambda = 5$, the null space of $A - \lambda I = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}$ is $\operatorname{Span}\left(\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \right)$.

It is easy to see that $\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is linearly independent and so forms a basis of $\mathbb{R}^2$. So for $Q = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, we have $Q^{-1}AQ = D$ with $D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ being diagonal. It is also easy to compute that $Q^{-1} = \begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{pmatrix}$.

Then for each $n \in \mathbb{Z}^+$ we have

$$A^n = (QDQ^{-1})^n = QD^nQ^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2(-1)^n + 5^n & 2(5^n - (-1)^n) \\ 5^n - (-1)^n & 2 \cdot 5^n + (-1)^n \end{pmatrix}$$

Note

Expression also holds for non-positive integer n with an easy proof.

When A is not diagonalizable, we may still use Cayley–Hamilton to simplify the computation, although it would then be much complicate than the diagonalizable case.

5.2.8. Suppose that $A \in M_{n\times n}(\mathbb{F})$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n-1$. Prove that A is diagonalizable.

Idea: To show that A is diagonalizable, we can check if the algebraic multiplicities of its eigenvalues are the same as the geometric multiplicities. Since $\dim(E_{\lambda_1}) = n-1$ is *almost* the same as $\dim(\mathbb{F}^n) = n$, there is little room for other eigenvalues.

Solution: Let $\gamma_A(\lambda), m_A(\lambda)$ are the geometric and algebraic multiplicity of an eigenvalue λ of A respectively.

Trivially, $1 \le \dim(E_{\lambda_2})$, so $n = (n-1) + 1 \le \dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) = \gamma_A(\lambda_1) + \gamma_A(\lambda_2) \le m_A(\lambda_1) + m_A(\lambda_2) \le \dim(\mathbb{F}^n) = n$. This implies that all inequalities are equalities, and so $\gamma_A(\lambda_1) = m_A(\lambda_1)$, $\gamma_A(\lambda_2) = m_A(\lambda_2)$, and $m_A(\lambda_1) + m_A(\lambda_2) = n$.

As $m_A(\lambda) \ge 1$ for every eigenvalues of A and the algebraic multiplicities of the eigenvalues of A sum to $\dim(\mathbb{F}^n) = n$, we can see that λ_1, λ_2 are the only eigenvalues of A. Since for each eigenvalue λ of A we have $m_A(\lambda) = \gamma_A(\lambda)$, A is diagonalizable.

5.2.13. Let $A \in M_{n\times n}(\mathbb{F})$. For any eigenvalue λ of A and A^T, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^T, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.
(b) Prove that for any eigenvalue λ, $\dim(E_\lambda) = \dim(E'_\lambda)$.
(c) Prove that if A is diagonalizable, then A^T is also digonalizable.

Solution:

(a) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Then $A^\mathsf{T} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. It is easy to see that A and A^T have a unique eigenvalue $\lambda = 1$, and $E_1 = \operatorname{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \right) \ne \operatorname{Span}\left(\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \right) = E'_1$.

(b) By a theorem in MATH1030, we have $\operatorname{rank}(A - \lambda I) = \operatorname{rank}((A-\lambda I)^\mathsf{T}) = \operatorname{rank}(A^\mathsf{T} - \lambda I)$, so $\dim(E_\lambda) = \operatorname{nullity}(A - \lambda I) = n - \operatorname{rank}(A - \lambda I) = n - \operatorname{nullity}(A^\mathsf{T} - \lambda I) = \operatorname{nullity}(A^\mathsf{T} - \lambda I) = \dim(E'_\lambda)$.

(c) Suppose A is diagonalizable. Then A has full eigenvalues, and for each eigenvalue λ we have $\gamma_A(\lambda) = m_A(\lambda)$. Since $\det(A - tI) = \det(A^\mathsf{T} - tI)$ for all $t \in \mathbb{F}$, A and A^T have the same characteristic polynomial. This implies that A^T also has full eigenvalues, and $m_A(\lambda) = m_{A^\mathsf{T}}(\lambda)$ for each eigenvalue λ.

By the previous part, $\gamma_A(\lambda) = \dim(E_\lambda) = \dim(E'_\lambda) = \gamma_{A^\mathsf{T}}(\lambda)$ for each eigenvalue λ. So $\gamma_{A^\mathsf{T}}(\lambda) = m_{A^\mathsf{T}}(\lambda)$ for each eigenvalue λ. This implies that A^T is diagonalizable.

Note

For part (b), we can also show that $\operatorname{rank}(A) = \operatorname{rank}(A^\mathsf{T})$ without going through the theorem from MATH1030, but that may require using concepts from later lectures (namely inner product and norm). Please send us an email if you have a good proof without using them.

5.2.18. (a) Prove that if T and U are simultaneously diagonalizable operators, then T and U commute.

(b) Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

Solution: We assume that $\dim(V) = n$, as in the definition for simultaneously diagonalizability.

(a) Since T, U are simultaneously diagonalizable, there exists an ordered basis β such that $[T]_\beta$ and $[U]_\beta$ are both diagonal. We may assume that the matrices are $[T]_\beta = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$, $[U]_\beta = \begin{pmatrix} f_1 & & \\ & \ddots & \\ & & f_n \end{pmatrix}$. Then $[TU]_\beta = [T]_\beta[U]_\beta = \begin{pmatrix} d_1 f_1 & & \\ & \ddots & \\ & & d_n f_n \end{pmatrix} = \begin{pmatrix} f_1 d_1 & & \\ & \ddots & \\ & & f_n d_n \end{pmatrix} = [U]_\beta [T]_\beta = [UT]_\beta$. This implies that $TU = UT$, and so T, U commute.

(b) Let $\alpha = \{ e_1, \dots, e_n \}$ be the standard basis of $\mathbb{F}^n$. Then $[\mathsf{L}_A]_\alpha = A$, $[\mathsf{L}_B]_\alpha = B$. Since A, B are simultaneously diagonalizable, there exists invertible $Q \in M_{n\times n}(\mathbb{F})$ such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are both diagonal.

Let $\beta = \{ Qe_1, \dots, Qe_n \}$. Since Q is invertible, β is an ordered basis of $\mathbb{F}^n$, and $[\mathrm{Id}]_\beta^\alpha = Q$. So $[\mathsf{L}_A]_\beta = [\mathrm{Id}]_\alpha^\beta [\mathsf{L}_A]_\beta [\mathrm{Id}]_\beta^\alpha = Q^{-1}AQ$ and $[\mathsf{L}_B]_\beta = Q^{-1}BQ$ are both diagonal. This implies that $\mathsf{L}_A, \mathsf{L}_B$ are simultaneously diagonalizable operators on $\mathbb{F}^n$, and thus by the previous part they commute. Hence $AB = [\mathsf{L}_A]_\alpha [\mathsf{L}_B]_\alpha = [\mathsf{L}_A \mathsf{L}_B]_\alpha = [\mathsf{L}_B \mathsf{L}_A]_\alpha = [\mathsf{L}_B]_\alpha [\mathsf{L}_A]_\alpha = BA$. So A, B commute.

Note

You can also work on the matrices directly in the same way part (a) is done rather than going through L_A and L_B.

Practice Problems

5.1.1. Label the following statements as true or false.

(a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

(b) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.

(c) There exists a square matrix with no eigenvectors.

(d) Eigenvalues must be nonzero scalars.

(e) Any two eigenvectors are linearly independent.

(f) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

(g) Linear operators on infinite-dimensional vector spaces never have eigenvalues.

(h) An $n \times n$ matrix A with entries from a field $\mathbb{F}$ is similar to a diagonal matrix if and only if there is a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

(i) Similar matrices always have the same eigenvalues.

(j) Similar matrices always have the same eigenvectors.

(k) The sum of two eigenvectors of an operator T is always an eigenvector of T.

Solution:

(a) False

(b) True

(c) True

(d) False

(e) False

(f) False

(g) False

(h) True

(i) True

(j) False

(k) False

5.1.6. Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$.

Solution: Let $\lambda \in \mathbb{F}$ be an eigenvalue of T. Then there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$. So $[T]_\beta[v]_\beta = [Tv]_\beta = [\lambda v]_\beta = \lambda[v]_\beta$. As $v \neq 0_V$, $[v]_\beta \neq 0_{\mathbb{F}^n}$. So λ is an eigenvalue of $[T]_\beta$ (as witnessed by $[v]_\beta$).

Let $\lambda \in \mathbb{F}$ be an eigenvalue of $[T]_\beta$. Then there exists a nonzero $x \in \mathbb{F}^n$ such that $[T]_\beta x = \mathsf{L}_{[T]_\beta} x = \lambda x$. Since V is finite-dimensional, we may assume that $\beta = \{\, v_1, \ldots, v_n \,\}$ for some $n \in \mathbb{N}$. Let $v = \sum_{i=1}^n x_i v_i$ where $x = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix}^\mathsf{T}$. Then $v \in V$ and $[v]_\beta = x$. This implies that $[Tv]_\beta = [T]_\beta[v]_\beta = [T]_\beta x = \lambda x = []\lambda v]_\beta$, so $Tv = \lambda v$. As $x \neq 0_{\mathbb{F}^n}$, $v \neq 0_V$. Hence λ is an eigenvalue of T (as witnessed by v).

Hence λ is an eigenvalue of T if and only if it is an eigenvalue of $[T]_\beta$.

5.1.7. Let T be a linear operator on a finite-dimensional vector space V.

(a) Prove that if β and γ are two ordered bases for V, then $\det([T]_\beta) = \det([T]_\gamma)$.

(b) Prove that T is invertible if and only if $\det(T) \neq 0$.

(c) Prove that if T is invertible, then $\det(T^{-1}) = [\det(T)]^{-1}$.

(d) Prove that if U is also a linear operator on V, then $\det(TU) = \det(T)\det(U)$.

(e) Prove that $\det(T - \lambda \mathrm{Id}_V) = \det([T]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

Solution:

(a) Let $Q = [\mathrm{Id}_V]_\beta^\gamma$. Then Q is invertible. So by the property of determinant, $\det([T]_\beta) = \det([\mathrm{Id}_V]_\gamma^\beta[T]_\gamma[\mathrm{Id}_V]_\beta^\gamma) = \det(Q^{-1}[T]_\gamma Q) = \det(Q)^{-1}\det([T]_\gamma)\det(Q) = \det([T]_\gamma)$

(b) Let β be an ordered basis of V. Then $\det(T) = \det([T]_\beta)$. By the property of matrix representation, T is invertible if and only if $[T]_\beta$ is, which holds if and only if $\det([T]_\beta) \neq 0$. So T is invertble if and only if $\det(T) \neq 0$.

(c) Suppose T is invertible. Then T^{-1} exists and $TT^{-1} = \mathrm{Id}_V$. Let β be an ordered basis. Then $1 = \det([\mathrm{Id}_V]_\beta) = \det([TT^{-1}]_\beta) = \det([T]_\beta[T^{-1}]_\beta) = \det([T]_\beta)\det([T^{-1}]_\beta) = \det(T)\det(T^{-1})$. So $\det(T^{-1}) = [\det(T)]^{-1}$.

(d) Let β be an ordered basis of V. Then $\det(TU) = \det([TU]_\beta) = \det([T]_\beta[U]_\beta) = \det([T]_\beta)\det([U]_\beta) = \det(T)\det(U)$.

(e) Let $\lambda \in \mathbb{F}$ and β be an ordered basis for V. Then $[T - \lambda\mathrm{Id}_V]_\beta = [T]_\beta - [\lambda\mathrm{Id}_V]_\beta = [T]_\beta - \lambda[\mathrm{Id}_V]_\beta = [T]_\beta - \lambda I$. Hence $\det(T - \lambda\mathrm{Id}_V) = \det([T - \lambda\mathrm{Id}_V]_\beta) = \det([T]_\beta - \lambda I)$.

5.1.8. (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(c) State and prove results analogous to (a) and (b) for matrices.

Solution:

(a) Suppose T is invertible. Then for all $v \neq 0$ we have $Tv \neq 0 = 0 \cdot v$. This implies that 0 is not an eigenvalue of T.

Suppose T is not invertible. Since the vector space V is finite-dimensional, this implies that T is not one-to-one. So there exists nonzero $v \in \mathsf{N}(T)$, or equivalently $Tv = 0 = 0 \cdot v$. Hence 0 is an eigenvalue of T (with an eigenvector v).

Thus T is invertible if and only if 0 is not an eigenvalue of T.

(b) Suppose λ is an eigenvalue of T. Since T is invertible, $\lambda \neq 0$. Also, by the definition of eigenvalue, there exists nonzero $v \in V$ such that $Tv = \lambda v$, so $T^{-1}v = T^{-1}(\lambda^{-1}\lambda v) = \lambda^{-1}T^{-1}(Tv) = \lambda^{-1}v$. Since v is nonzero, this implies that λ^{-1} is an eigenvalue of T^{-1}.

Suppose λ^{-1} is an eigenvalue of T^{-1}. Since T is invertible, T^{-1} is also invertible. By the previous part, λ^{-1} is nonzero. By the previous proof, $\lambda = \left(\lambda^{-1}\right)^{-1}$ is an eigenvalue of $\left(T^{-1}\right)^{-1} = T$.

So λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Note

See also the spectral mapping theorem.

(c) The analogous results are:

- A matrix $A \in M_{n\times n}(\mathbb{F})$ is invertible if and only if 0 is not an eigenvalue of A.
- If $A \in M_{n\times n}(\mathbb{F})$ is invertible, then $\lambda \in \mathbb{F}$ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

The proofs are easy corollaries of the previous parts: let α be the standard ordered basis of $\mathbb{F}^n$. Then

- A is invertible if and only if L_A is invertible, which by part (a) holds if and only if 0 is not an eigenvalue of L_A, which by Question 5.1.6 holds if and only if 0 is not an eigenvalue of $A = [\mathsf{L}_A]_\alpha$.
- Suppose A is invertible. Then by Question 5.1.6, λ is an eigenvalue of $A = [\mathsf{L}_A]_\alpha$ if and only if it is an eigenvalue of L_A, which by part (b) holds if and only if λ^{-1} is an eigenvalue of $\mathsf{L}_\alpha{}^{-1}$, which again by Question 5.1.6 holds if and only if λ^{-1} is an eigenvalue of $[\mathsf{L}_A{}^{-1}]_\alpha = ([\mathsf{L}_A])^{-1} = A^{-1}$.

5.1.11. (a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

Solution:

(a) Suppose $A \in M_{n\times n}(\mathbb{F})$ is similar to λI for some scalar $\lambda \in \mathbb{F}$. Then there exists invertible $B \in M_{n\times n}(\mathbb{F})$ such that $B^{-1}AB = \lambda I$, so $A = B(\lambda I)B^{-1} = \lambda BIB^{-1} = \lambda I$.

(b) Suppose $A \in M_{n\times n}(\mathbb{F})$ is a diagonalizable matrix which has only one eigenvalue. Then there exists invertible $Q \in M_{n\times n}(\mathbb{F})$ such that $D = Q^{-1}AQ$ is diagonal. Let $\alpha = \{ e_1, \ldots, e_n \}$ be the standard ordered basis of $\mathbb{F}^n$. Since D is diagonal, we have $De_i = D_{ii}e_i$ with $e_i \neq 0$, so D has n eigenvalues (counted with multiplicity) $D_{11}, \ldots, D_{nn}$. Since A is similar to D, they have the same eigenvalues. Since A has only one eigenvalue, this implies that $D_{11} = \ldots = D_{nn} = \lambda$ for some $\lambda \in \mathbb{F}$. So $D = \lambda I$ is a scalar matrix. In particular, A is similar to a scalar matrix, so by part (a) A itself is a scalar matrix.

5.1.14. For any square matrix A, prove that A and A^T have the same characteristic polynomial (and hence the same eigenvalues).

Solution: Assume that $A \in M_{n\times n}(\mathbb{F})$.

By the property of characteristic polynomial, $\det(A - tI), \det((A - tI)^\mathsf{T} = \det(A^\mathsf{T} - tI)$ are elements in the polynomial ring $\mathbb{F}[t]$ in t. Let $K = \mathrm{Frac}(\mathbb{F}[t])$, the field of fractions of $\mathbb{F}[t]$. Then $K \supseteq \mathbb{F}[t] \supseteq \mathbb{F}$, so $A - tI, A^\mathsf{T} - tI \in M_{n\times n}(K)$. By Theorem 4.8 in textbook, $\det(A-tI) = \det(A^\mathsf{T}-tI)$ in K. As $K \supseteq \mathbb{F}[t]$ and $\det(A-tI), \det(A^\mathsf{T}-tI) \in \mathbb{F}[t]$, $\det(A-tI) = \det(A^\mathsf{T}-tI)$ in $\mathbb{F}[t]$. This implies that A and A^T have the same characteristic polynomial.

As eigenvalues are exactly the roots of the characteristic polynomial, A and A^T have the same eigenvalues.

Note

Note that we do not have information on the scalar field.

5.1.16. (a) Prove that similar matrices have the same trace.

(b) How would you define the trace of a linear operator on a finite-dimensional vector space? Justify that your definition is well-defined.

Solution:

(a) Let $A, B \in M_{n\times n}(\mathbb{F})$ be similar. Then there exists invertible $Q \in M_{n\times n}(\mathbb{F})$ such that $B = Q^{-1}AQ$. Then by the property of trace, we have $\operatorname{tr}(B) = \operatorname{tr}(Q^{-1}AQ) = \operatorname{tr}(AQQ^{-1}) = \operatorname{tr}(A)$

(b) For a linear operator T on a finite dimensional vector space V define $\operatorname{tr}(T) = \operatorname{tr}([T]_\beta)$ where β is an ordered basis of V. We now show that this definition is well-defined. Let α, β be ordered basis of V. Then $[T]_\alpha = [\mathrm{Id}]_\beta^\alpha [T]_\beta [\mathrm{Id}]_\alpha^\beta = \left([\mathrm{Id}]_\alpha^\beta\right)^{-1}[T]_\beta[\mathrm{Id}]_\alpha^\beta$, so $[T]_\alpha$ and $[T]_\beta$ are similar matrices. By the previous part, $\operatorname{tr}([T]_\alpha) = \operatorname{tr}([T]_\beta)$. Hence $\operatorname{tr}(T)$ is independent of the choice of the ordered basis, and thus is well-defined.

5.1.21. Let A and $f(t)$ be as in Question 5.1.20.

(a) Prove that $f(t) = (A_{11} - t)(A_{22} - t)\dots(A_{nn} - t) + q(t)$, where $q(t)$ is a polynomial of degree at most $n - 2$.

(b) Show that $\operatorname{tr}(A) = (-1)^{n-1}a_{n-1}$.

Solution:

(a) We will use the following lemma:

> Let $B \in M_{n\times n}(\mathbb{F}[t])$ be such that for some $k \le n^2$ there are at most k entries of B are polynomials of degree 1 and all other entries are scalars. Then $\det(B)$ is a polynomial of degree at most k.

To show the proposition, we will use induction on the size of the matrix n.

For the case $n = 1$, we have $A = \begin{pmatrix} A_{11} \end{pmatrix}$ for some scalar $A_{11} \in \mathbb{F}$. Then $f(t) = \det(A - tI) = A_{11} - t + q(t)$ with $q(t) = 0$ is a polynomial of degree at most -1.

Suppose for some integer $k \in \mathbb{Z}^+$ the proposition holds for all matrices of size $k \times k$, and $A \in M_{(k+1)\times(k+1)}(\mathbb{F})$. Then $f(t) = \det(A - tI) = \det \begin{pmatrix} A_{11} - t & * & & * \\ * & A_{22} - t & & * \\ & & \ddots & \\ A_{k+1,1} & A_{k+1,2} & \dots & A_{k+1,k+1} - t \end{pmatrix}$. Expanding along the last row, we have $f(t) = \sum_{i=1}^{k}(-1)^{k+1+i}A_{k+1,i}M_{k+1,i} + (-1)^{2(k+1)}(A_{k+1,k+1} - t)\det \begin{pmatrix} A_{11} - t & * & & * \\ * & A_{22} - t & & * \\ & & \ddots & \\ A_{k+1,1} & A_{k+1,2} & \dots & A_{kk} - t \end{pmatrix}$ where M_{ij} is the minor of $A - tI$ at (i, j). By induction assumption, $\det \begin{pmatrix} A_{11} - t & * & & * \\ * & A_{22} - t & & * \\ & & \ddots & \\ A_{k+1,1} & A_{k+1,2} & \dots & A_{kk} - t \end{pmatrix} = (A_{11} - t)\dots(A_{kk} - t) + q'(t)$ where q' is a polynomial of degree at most $k - 2$. Also, as each of the submatrices contains only $k + 1 - 2 = k - 1$ entries that are polynomials of degree 1, using the lemma we can see that each of $M_{k+1,i}$ for $i \in \{ 1, \dots, k \}$ is a polynomial of degree at most $k - 1$. This implies that $f(t) = (A_{11} - t)\dots(A_{k+1,k+1} - t) + q(t)$ where $q(t) = (A_{k+1,k+1} - t)q'(t) + \sum_{i=1}^{k}(-1)^{k+1+i}A_{k+1,i}M_{k+1,i}$ is a polynomial of degree at most $k - 1$.

By induction, the proposition holds for all $n \in \mathbb{Z}^+$.

It remains to show the lemma. To do so, we will again use induction on n. It is easy to see that the proposition holds for the base case $n = 1$.

Suppose for some integer $m \in \mathbb{Z}^+$ the proposition holds for all matrices of size $m \times m$, and $B \in M_{(m+1)\times(m+1)}(\mathbb{F}[t])$ has k entries of polynomial of degree 1. Then $\det(B) = \sum_{i=1}^{m+1}(-1)^{1+i}B_{1i}M_{1i}$ where $M_{ij} = \det(B_{-i,-j})$ is the minor of B at (i, j) and $B_{-i,-j} \in M_{m\times m}(\mathbb{F}[t])$ is the submatrix obtained after removing ith row and jth column from B. Then for each $i \in \{ 1, \dots, m + 1 \}$,

- If B_{1i} is a scalar, then $B_{-1,-i}$ contains at most $\min(k, m^2)$ entries of polynomial of degree 1. By induction assumption, $M_{1j} = \det(B_{-1,-i})$ is a polynomial of degree at most $\min(k, m^2) \le k$, and so $(-1)^{1+i}B_{1i}M_{1i}$ is a polynomial of degree at most k.

- If B_{1i} is a polynomial of degree 1, then then $B_{-1,-i}$ contains at most $\min(k-1, m^2)$ entries of polynomial of degree 1. By induction assumption, $M_{1j} = \det(B_{-1,-i})$ is a polynomial of degree at most $\min(k-1, m^2) \leq k-1$, and so $(-1)^{1+i}B_{1i}M_{1i}$ is a polynomial of degree at most k.

Hence $\det(B)$ is a sum of polynomials of degree at most k and so is a polynomial of degree at most k.

As k is arbitrary, by induction the lemma holds for all $n \in \mathbb{Z}^+$.

(b) Let $p(t) = (A_{11} - t)\dots(A_{nn} - t)$. By Vieta's formula, we have that p has leading coefficient $(-1)^n$, and $(-1)^{n+1}\operatorname{tr}(A) = -(-1)^n \sum_{i=1}^n A_{ii}$ is the coefficient of t^{n-1} term in the polynomial p. Since $\deg q \leq n-2$, it is also the coefficient of t^{n-1} term in the polynomial $f(t) = p(t) + q(t) = (A_{11} - t)\dots(A_{nn} - t) + q(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \dots$. So $(-1)^{n+1}\operatorname{tr}(A) = a_{n-1}$, or $\operatorname{tr}(A) = (-1)^{n-1}a_{n-1}$.

Note

The bound k used in the lemma in part (a) is very loose but is already sufficient for our use. You can also refine and generalize the lemma yourself.

5.2.1. Label the following statements as true or false.

(a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(b) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

(c) If A is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

(d) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

(e) Let $A \in M_{n\times n}(\mathbb{F})$ and $\beta = \{ v_1, v_2, \dots, v_n \}$ be an ordered basis for $\mathbb{F}^n$ consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose jth column is v_j $(1 \leq j \leq n)$, then $Q^{-1}AQ$ is a diagonal matrix.

(f) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ

(g) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

(h) If a vector space is the direct sum of subspaces $W_1, W_2, \dots, W_k$, then $W_i \cap W_j = \{0\}$ for $i \neq j$.

(i) If $V = \sum_{i=1}^k W_i$ and $W_i \cap W_j = \{0\}$ for $i \neq j$, then $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$.

Solution:

(a) False

(b) False

(c) False. Note that $0 \in E_\lambda$

(d) True

(e) True

(f) True

(g) True

(h) True. See also the next statement

(i) False. Note that a stronger condition is needed

5.2.10. Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ and corresponding multiplicities $m_1, m_2, \dots, m_k$. Suppose that β is a basis for V such that $[T]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[T]_\beta$ are $\lambda_1, \lambda_2, \dots, \lambda_k$ and that each λ_i occurs m_i times $(1 \leq i \leq k)$.

Solution: Assume that $\dim(V) = |\,\beta\,| = n$.

We evaluate the characteristic polynomial of T using the basis β. By definition, the characteristic polynomial is $p(t) = \det([T]_\beta - tI)$. Since $[T]_\beta$ is upper triangular, so is $[T]_\beta - tI$. Hence $\det([T]_\beta - tI) = \prod_{i=1}^n ([T]_\beta - tI)_{ii} = \prod_{i=1}^n (([T]_\beta)_{ii} - t)$. By assumption, the roots of $p(t)$ are $\lambda_1, \dots, \lambda_k$ with multiplicities $m_1, \dots, m_k$, so $p(t)$ is a multiple of $\prod_{i=1}^k (t - \lambda_i)^{m_i}$. Hence

$\prod_{i=1}^{n}(([T]_\beta)_{ii}-t) = p(t) = q(t)\prod_{i=1}^{k}(t-\lambda_i)^{m_i}$. By assumption, $\sum_{i=1}^{k} m_i = n$, so $n = \deg\prod_{i=1}^{n}(([T]_\beta)_{ii}-t) = \deg q + \deg\prod_{i=1}^{k}(t-\lambda_i)^{m_i} = \deg q + \sum_{i=1}^{k} m^i = n + \deg q$, $\deg q = 0$, and thus q is a scalar. This implies that $([T]_\beta)_{11}, \dots, ([T]_\beta)_{nn}$ are $\lambda_1, \dots, \lambda_k$ each occurs $m_1, \dots, m_k$ times respectively.

5.2.11. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \dots, m_k$. Prove the following statements.

(a) $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i\lambda_i$

(b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\dots(\lambda_k)^{m_k}$

Solution: Let $Q \in M_{n\times n}(\mathbb{R})$ be invertible such that $Q^{-1}AQ$ is upper triangular. Let $\alpha = \{ e_1, \dots, e_n \}$ be the standard basis of $\mathbb{F}^n$, and $\beta = \{ Qe_1, \dots, Qe_n \}$. Then β is a basis, $[\mathsf{L}_A]_\alpha = A$, $[\mathrm{Id}]_\beta^\alpha = Q$, and so $[T]_\beta = Q^{-1}AQ$ is upper triangular. By Question 5.2.10, the diagonal entries of $[T]_\beta$ are $\lambda_1, \dots, \lambda_k$ each occurs $m_1, \dots, m_k$ times respectively.

(a) By the result of Question 5.1.16, $\operatorname{tr}(A) = \operatorname{tr}([T]_\beta) = \sum_{i=1}^{n}([T]_\beta)_{ii} = \sum_{i=1}^{k} m_i\lambda_i$.

(b) By the property of determinant, $\det(A) = \det([T]_\beta) = \prod_{i=1}^{n}([T]_\beta)_{ii} = \prod_{i=1}^{k}\lambda_i^{m_i}$.

5.2.12. Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Solution:

(a) Let λ be an eigenvalue of T, and $v \in E_\lambda(T)$ be in the eigenspace of T corresponding to the eigenvalue λ. Then $T(v) = \lambda v$, so $T^{-1}(v) = \lambda^{-1}T^{-1}(\lambda v) = \lambda^{-1}T^{-1}(Tv) = \lambda^{-1}v$. This implies that $v \in E_{\lambda^{-1}}(T^{-1})$. As v is arbitrary, $E_\lambda(T) \subseteq E_{\lambda^{-1}}(T^{-1})$.

Since T is invertible, T^{-1} is also invertble with $\left(T^{-1}\right)^{-1} = T$. So by the same argument and the result of Question 5.1.8, we have $E_{\lambda^{-1}}(T^{-1}) \subseteq E_{(\lambda^{-1})^{-1}}(\left(T^{-1}\right)^{-1}) = E_\lambda(T)$. This implies that $E_\lambda(T) = E_{\lambda^{-1}}(T^{-1})$.

Thus the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}

(b) Suppose T is diagonalizable. Let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be all of the distinct eigenvalues of T. Then $\gamma_T(\lambda_i) = m_T(\lambda_i)$ for each i, and $\sum_{i=1}^{k} m_T(\lambda_i) = \dim(V)$. By part (b) of Question 5.1.8, the eigenvalues of T^{-1} are exactly ${\lambda_1}^{-1}, \dots, {\lambda_k}^{-1}$. By part (a), $E_\lambda(T) = E_{\lambda^{-1}}(T^{-1})$, so $\gamma_T(\lambda) = \gamma_{T^{-1}}(\lambda^{-1})$. This implies that $\dim(V) \geq \sum_{i=1}^{k} m_{T^{-1}}({\lambda_i}^{-1}) \geq \sum_{i=1}^{k}\gamma_{T^{-1}}({\lambda_i}^{-1}) = \sum_{i=1}^{k}\gamma_T(\lambda_i) = \sum_{i=1}^{k} m_T(\lambda_i) = \dim(V)$. Hence $m_{T^{-1}}({\lambda_i}^{-1}) = \gamma_{T^{-1}}({\lambda_i}^{-1})$ for each i. So T^{-1} is invertible.

5.2.17. (a) Prove that if T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, then the matrices $[T]_\beta$ and $[U]_\beta$ are simultaneously diagonalizable for any ordered basis β.

(b) Prove that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable linear operators.

Solution:

(a) Since T, U are simultaneously diagonalizable, there exists an ordered basis α such that $[T]_\alpha$ and $[U]_\alpha$ are diagonal.

Let β be an ordered basis. Then $[T]_\beta = [\mathrm{Id}]_\alpha^\beta[T]_\alpha[\mathrm{Id}]_\beta^\alpha = Q[T]_\alpha Q^{-1}$ and $[U]_\beta = Q[U]_\alpha Q^{-1}$ with $Q = [\mathrm{Id}]_\alpha^\beta$. By assumption, $[T]_\alpha = Q^{-1}[T]_\beta Q$ and $[U]_\alpha = Q^{-1}[U]_\beta Q$ are both diagonal, so by definition $[T]_\beta$ and $[U]_\beta$ are simultaneously diagonalizable.

(b) Suppose A, B are simultaneously diagonalizable. Then there exists invertible matrix Q such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are both diagonal.

Let α be the standard basis. Then $[\mathsf{L}_A]_\alpha = A$ and $[\mathsf{L}_B]_\alpha = B$. Let $\beta = \{ Qe_1, \dots, Qe_n \}$. Since Q is invertible, β is a basis, and $[\mathrm{Id}]_\beta^\alpha = Q$. So by assumption, $Q^{-1}AQ = [\mathrm{Id}]_\alpha^\beta[\mathsf{L}_A]_\alpha[\mathrm{Id}]_\beta^\alpha = [\mathrm{Id} \circ \mathsf{L}_A \circ \mathrm{Id}]_\beta = [\mathsf{L}_A]_\beta$ and $Q^{-1}BQ = [\mathsf{L}_B]_\beta$ are both diagonal. By definition, L_A and L_B are simultaneously diagonalizable.