MATH2040 Homework 3
Reference Solution

2.2.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{\ (1,1,0), (0,1,1), (2,2,3)\ \}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{\ (1,2), (2,3)\ \}$, compute $[T]_\alpha^\gamma$.

Idea: The general approach to compute $[T]_\alpha^\beta$ is to compute $T(v)$ for all $v \in \alpha$, then represent these vectors in β-coordinate. The matrix $[T]_\alpha^\beta$ is then formed by arranging the coefficients appropriately.

Solution:

$$\begin{aligned} T(1,0) &= (1,1,2) = -\frac{1}{3} \cdot (1,1,0) + 0 \cdot (0,1,1) + \frac{2}{3} \cdot (2,2,3) \\ T(0,1) &= (-1,0,1) = -1 \cdot (1,1,0) + 1 \cdot (0,1,1) + 0 \cdot (2,2,3) \end{aligned}$$

Hence $[T]_\beta^\gamma = \begin{pmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{pmatrix}$.

$$\begin{aligned} T(1,2) &= (-1,1,4) = -\frac{7}{3} \cdot (1,1,0) + 2 \cdot (0,1,1) + \frac{2}{3} \cdot (2,2,3) \\ T(2,3) &= (-1,2,7) = -\frac{11}{3} \cdot (1,1,0) + 3 \cdot (0,1,1) + \frac{4}{3} \cdot (2,2,3) \end{aligned}$$

Hence $[T]_\alpha^\gamma = \begin{pmatrix} -7/3 & -11/3 \\ 2 & 3 \\ 2/3 & 4/3 \end{pmatrix}$

2.2.5. Let

$$\begin{aligned} \alpha &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \\ \beta &= \{\ 1, x, x^2\ \} \\ \gamma &= \{1\} \end{aligned}$$

(a) Define $T : M_{2\times 2}(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ by $T(A) = A^\mathsf{T}$. Compute $[T]_\alpha$

(b) Define $T : \mathsf{P}_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T(f) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$. Compute $[T]_\beta^\alpha$

(c) Define $T : M_{2\times 2}(\mathbb{F}) \to \mathbb{F}$ by $T(A) = \mathrm{tr}(A)$. Compute $[T]_\alpha^\gamma$

(d) Define $T : \mathsf{P}_2(\mathbb{R}) \to \mathbb{R}$ by $T(f) = f(2)$. Compute $[T]_\beta^\gamma$.

(e) If $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$, compute $[A]_\alpha$

(f) If $f(x) = 3 - 6x + x^2$, compute $[f]_\beta$.

(g) For $\alpha \in \mathbb{F}$, compute $[a]_\gamma$.

Solution:

(a) $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Hence $[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b)

$$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(c) $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \cdot 1$, $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$, $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$, $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot 1$. Hence $[T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$

(d) $T(1) = 1 \cdot 1$, $T(x) = 2 \cdot 1$, $T(x^2) = 4 \cdot 1$. Hence $[T]_\beta^\alpha = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$

(e) $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so $[A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}$

(f) $f = 3 - 6x + x^2 = 3 \cdot 1 - 6 \cdot x + 1 \cdot x^2$, so $[f]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$

(g) $a = a \cdot 1$, so $[a]_\gamma = \begin{pmatrix} a \end{pmatrix}$

2.2.9. Let V be the vector space of complex numbers over the field $\mathbb{R}$. Define $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Prove that T is linear, and compute $[T]_\beta$, where $\beta = \{ 1, i \}$.

Solution: We first show that T is linear.

Let $z_1, z_2 \in \mathbb{C}$, $a \in \mathbb{R}$. Then there exists $r_1, r_2, m_1, m_2 \in \mathbb{R}$ such that $z_1 = r_1 + im_1$, $z_2 = r_2 + im_2$. So

- $T(z_1 + z_2) = \overline{(r_1 + r_2) + i(m_1 + m_2)} = (r_1 - im_1) + (r_2 - im_2) = T(z_1) + T(z_2)$
- $T(az_1) = \overline{(ar_1) + i(am_2)} = a(r_1 - im_1) = aT(z_1)$

As z_1, z_2, a are arbitrary, T is linear.

$$T(1) = 1 = 1 \cdot 1 + 0 \cdot i$$
$$T(i) = -i = 0 \cdot 1 - 1 \cdot i$$

Hence $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note

See also Question 2.1.38 in Homework 2.

2.2.13. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $\mathsf{R}(T) \cap \mathsf{R}(U) = \{0\}$, prove that $\{ T, U \}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

Solution: Since T, U are nonzero transformations, $\mathsf{R}(T) \neq \{0\}$ and $\mathsf{R}(U) \neq \{0\}$. If $T = U$, we would have $\{0\} = \mathsf{R}(T) \cap \mathsf{R}(U) = \mathsf{R}(T) \neq \{0\}$. This implies that $T \neq U$.

By the result of Question 1.5.9 in Homework 2, it suffices to show that T, U is not a multiple of the other.

Suppose otherwise that $T = \lambda U$ for some scalar λ. As $T \neq 0$ and $U \neq 0$, we must have $\lambda \neq 0$. So $\mathsf{R}(T) = \mathsf{R}(\lambda U) = \{ \lambda U(v) : v \in V \} = \{ U(\lambda v) : v \in V \} = \{ U(v) : v \in V \} = \mathsf{R}(U)$, and thus $\{0\} = \mathsf{R}(T) \cap \mathsf{R}(U) = \mathsf{R}(U) \neq \{0\}$. Contradiction arises. Hence T is not a scalar multiple of U. By symmetry, U is not a scalar multiple of T.

Therefore $\{ T, U \}$ is linearly independent.

Note

You can also show this by noting that a linear relation on T, U gives the same linear relation on $T(v), U(v)$ for all $v \in V$, and work on this relation (on vectors) instead.

2.2.14. Let $V = \mathsf{P}(\mathbb{R})$, and for $j \geq 1$ define $T_j(f) = f^{(j)}$ where $f^{(j)}$ is the jth derivative of f. Prove that the set $\{ T_1, T_2, \ldots, T_n \}$ is a linearly independent subset of $\mathcal{L}(V)$ for any positive integer n.

Solution: Let $n \in \mathbb{Z}^+$, $a_1, \ldots, a_n \in \mathbb{R}$ be such that $\sum_{i=1}^n a_i T_i = 0$. Then for all $p \in V$, we have $\sum_{i=1}^n a_i T_i(p) = \sum_{i=1}^n a_i p^{(i)} = 0$. In particular, $\sum_{i=1}^n a_i T(x^n) = \sum_{i=1}^n a_i \frac{n!}{(n-i)!} x^{n-i} = 0$. As the degree of $\{ x^{n-i} : i \in \{ 1, \ldots, n \} \}$ are all distinct, by comparing coefficients we have $a_1 \frac{n!}{(n-1)!} = \ldots = a_n \frac{n!}{n!} = 0$ and so $a_1 = \ldots = a_n = 0$ as none of $\frac{n!}{(n-i)!}$ is zero. This implies that $\{ T_1, \ldots, T_n \}$ is linearly independent.

As n is arbitrary, $\{ T_1, T_2, \ldots, T_n \}$ is a linearly independent subset of $\mathcal{L}(V)$ for every $n \in \mathbb{Z}^+$.

2.3.3. Let $g(x) = 3 + x$. Let $T : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ and $U : \mathsf{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f) = f'g + 2f \quad \text{and} \quad U(a + bx + cx^2) = (a + b, c, a - b)$$

Let β and γ be the standard ordered bases of $\mathsf{P}_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(a) Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h]_\beta$ and $[U(h)]_\gamma$. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

Solution: By assumption, $\beta = \{ 1, x, x^2 \}$ and $\gamma = \{ (1,0,0), (0,1,0), (0,0,1) \}$.

(a)

$$\begin{aligned} U(1) &= (1,0,1) = 1 \cdot (1,0,0) + 1 \cdot (0,0,1) \\ U(x) &= (1,0,-1) = 1 \cdot (1,0,0) - 1 \cdot (0,0,1) \\ U(x^2) &= (0,1,0) \end{aligned}$$

So $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

$$\begin{aligned} T(1) &= 2 = 2 \cdot 1 \\ T(x) &= 3x + 3 = 3 \cdot 1 + 3 \cdot x \\ T(x^2) &= 4x^2 + 6x = 6 \cdot x + 4 \cdot x^2 \end{aligned}$$

So $[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$.

$$\begin{aligned} UT(1) &= U(2) = (2,0,2) = 2 \cdot (1,0,0) + 2 \cdot (0,0,1) \\ UT(x) &= U(3 + 3x) = (6,0,0) = 6 \cdot (1,0,0) \\ UT(x^2) &= U(6x + 4x^2) = (6,4,-6) = 6 \cdot (1,0,0) + 4 \cdot (0,1,0) - 6 \cdot (0,0,1) \end{aligned}$$

So $[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$.

By computation we have $[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$, which is consistent with the result.

(b) $h = 3 - 2x + x^2 = 3 \cdot 1 - 2 \cdot x + 1 \cdot x^2$, so $[h]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

$U(h) = (1, 1, 5) = 1 \cdot (1, 0, 0) + 1 \cdot (0, 1, 0) + 5 \cdot (0, 0, 1)$, so $[U(h)]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$.

By computation we have $[U(h)]_\gamma = [U]_\beta^\gamma [h]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$, which is consistent with the result.

2.3.12. Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

(b) Prove that if UT is onto, then U is onto. Must T also be onto?

(c) Prove that if U and T are one-to-one and onto, then UT is also.

Solution:

(a) Suppose UT is one-to-one. Let $v \in \mathsf{N}(T)$. Then $T(v) = 0$, so $UT(v) = U(T(v)) = U(0) = 0$. As UT is one-to-one and linear, we must have $v = 0$. As v is arbitrary, this implies that $\mathsf{N}(T) = \{0\}$ and so T is one-to-one.

Consider the case where $V = W = Z$ is the real vector space of real sequences, U is the left-shift operator, and T is the right-shift operator as defined in Question 2.1.21 in Homework 2. Then $UT = \mathrm{Id}_V$ is one-to-one, but U is not one-to-one, as proven in Question 2.1.21.

(b) Suppose UT is onto. Let $z \in Z$. Since UT is onto, there exists $v \in V$ such that $z = UT(v) = U(T(v)) \in \mathsf{R}(U)$. As z is arbitrary, this implies that U is onto.

Consider the same example as in the previous part. Then $UT = \mathrm{Id}_V$ is onto, but T is not onto, as proven in Question 2.1.21.

(c) Let $v \in \mathsf{N}(UT)$. Then $0 = UT(v) = U(T(v))$. As U is one-to-one, this implies that $T(v) = 0$. As T is also one-to-one, $v = 0$. As v is arbitrary, this implies that UT is one-to-one.

Let $z \in Z$. Since U is onto, there exists $w \in W$ such that $U(w) = z$. Since T is onto, there exists $v \in V$ such that $T(v) = w$. So $z = U(w) = U(T(v)) = UT(v) \in \mathsf{R}(UT)$. As z is arbitrary, this implies that UT is onto.

Note

See the analog in set theory, which has essentially the same proof.

2.3.16. Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear.

(a) If $\mathrm{rank}(T) = \mathrm{rank}(T^2)$, prove that $\mathsf{R}(T) \cap \mathsf{N}(T) = \{0\}$. Deduce that $V = \mathsf{R}(T) \oplus \mathsf{N}(T)$.

(b) Prove that $V = \mathsf{R}(T^k) \oplus \mathsf{N}(T^k)$ for some positive integer k.

Idea:

(a) We want to show that $v = 0$ for each $v \in \mathsf{R}(T) \cap \mathsf{N}(T)$. Since $v \in \mathsf{R}(T)$, we must have $v = T(w)$ for some $w \in V$, and so $v = 0$ if and only if $w \in \mathsf{N}(T)$. Since $T^2(w) = T(v) = 0$, we know that w must be in $\mathsf{N}(T^2)$. So it suffices to show that $\mathsf{N}(T^2) \subseteq \mathsf{N}(T)$.

(b) By part (a), it suffices to show that $\mathrm{rank}(T^k) = \mathrm{rank}((T^k)^2) = \mathrm{rank}(T^{2k})$ for some positive integer k. To do so, we may as well consider the behavior of the integer sequence $(\mathrm{rank}(T^n))_{n \in \mathbb{N}}$. Since $0 \leq \mathrm{rank}(T^n) = \dim(\mathsf{R}(T^n)) \leq \dim(V) < \infty$, the sequence can take only finitely many values. So, if we can find some properties of this sequence, we may be able to find the appropriate k.

Solution:

(a) For each $v \in \mathsf{N}(T)$ we have $T(v) = 0$ and so $T^2(v) = T(0) = 0$, $v \in \mathsf{N}(T^2)$. This implies that $\mathsf{N}(T) \subseteq \mathsf{N}(T^2)$.

Since $\text{rank}(T) = \text{rank}(T^2)$, by dimension theorem we have $\dim \mathsf{N}(T) = \text{nullity}(T) = \dim(V) - \text{rank}(T) = \dim(V) - \text{rank}(T^2) = \text{nullity}(T^2) = \dim \mathsf{N}(T^2)$. Hence $\mathsf{N}(T) = \mathsf{N}(T^2)$.

Let $v \in \mathsf{N}(T) \cap \mathsf{R}(T)$. Then $T(v) = 0$ and there exists $w \in V$ such that $v = T(w)$. So $T^2(w) = T(T(w)) = T(v) = 0$, $w \in \mathsf{N}(T^2) = \mathsf{N}(T)$. Thus $v = T(w) = 0$. As v is arbitrary, this implies that $\mathsf{N}(T) \cap \mathsf{R}(T) = \{0\}$.

Trivially, $\mathsf{R}(T) + \mathsf{N}(T) \subseteq V$. By the result of Question 1.6.29(a) in Homework 2 and the dimension theorem, we have $\dim(\mathsf{R}(T) + \mathsf{N}(T)) = \dim(\mathsf{R}(T)) + \dim(\mathsf{N}(T)) - \dim(\mathsf{R}(T) \cap \mathsf{N}(T)) = \dim(\mathsf{R}(T)) + \dim(\mathsf{N}(T)) = \dim(V)$. Hence $V = \mathsf{R}(T) + \mathsf{N}(T)$. By definition, $V = \mathsf{R}(T) \oplus \mathsf{N}(T)$.

(b) Let $n \in \mathbb{N}$. Then for each $v \in \mathsf{R}(T^{n+1})$ there exists $w \in V$ such that $v = T^{n+1}(w) = T^n(T(w)) \in \mathsf{R}(T^n)$, and so $v \in \mathsf{R}(T^n)$. This implies that $\mathsf{R}(T^{n+1}) \subseteq \mathsf{R}(T^n)$, and so $\text{rank}(T^{n+1}) \leq \text{rank}(T^n)$. As $n \in \mathbb{N}$, the sequence $(\text{rank}(T^n))_{n\in\mathbb{N}}$ is non-increasing.

For each $n \in \mathbb{N}$, as $\mathsf{R}(T^n) \subseteq V$, we must have $0 \leq \text{rank}(T^n) \leq \dim(V) < \infty$. This implies that $\text{rank}(T^n) \in \{0, \ldots, \dim(V)\}$ for each $n \in \mathbb{N}$. As $\{0, \ldots, \dim(V)\}$ is a finite set, by pigeonhole principle there must exist $k \in \mathbb{N}$ and $l \in \mathbb{Z}^+$ such that $\text{rank}(T^k) = \text{rank}(T^{k+l})$. By the non-increasing property of the sequence, $\text{rank}(T^k) = \text{rank}(T^{k+1}) = \ldots = \text{rank}(T^{k+l})$, thus we may assume that $l = 1$.

Since $\text{rank}(T^{k+1}) = \text{rank}(T^k) \leq \dim(V) < \infty$ and $\mathsf{R}(T^{k+1}) \subseteq \mathsf{R}(T^k)$, we have $\mathsf{R}(T^k) = \mathsf{R}(T^{k+1})$.

Suppose $\mathsf{R}(T^m) = \mathsf{R}(T^{m+1})$ for some $m \in \mathbb{N}$. By the previous argument, we have $\mathsf{R}(T^{m+2}) \subseteq \mathsf{R}(T^{m+1})$.

Let $v \in \mathsf{R}(T^{m+1})$. Then there exists $w \in V$ such that $v = T^{m+1}(w) = T(T^m(w))$. As $T^m(w) \in \mathsf{R}(T^m) = \mathsf{R}(T^{m+1})$, there exists $u \in V$ such that $T^m(w) = T^{m+1}(u)$, so $v = T(T^m(w)) = T(T^{m+1}(u)) = T^{m+2}(u) \in \mathsf{R}(T^{m+2})$. As v is arbitrary, $\mathsf{R}(T^{m+1}) \subseteq \mathsf{R}(T^{m+2})$, and so $\mathsf{R}(T^{m+1}) = \mathsf{R}(T^{m+2})$.

By induction, $\mathsf{R}(T^k) = \mathsf{R}(T^{k+m})$ for all $m \in \mathbb{N}$. In particular, $\mathsf{R}(T^k) = \mathsf{R}(T^{2k})$, and so $\text{rank}(T^k) = \text{rank}(T^{2k}) = \text{rank}((T^k)^2)$.

By part (a), $V = \mathsf{R}(T^k) \oplus \mathsf{N}(T^k)$.

Note

Using monotone convergence theorem in part (b) on the bounded integer sequence $(\text{rank}(T^n))_{n\in\mathbb{N}}$ instead of pigeonhole principle gives you directly that $\text{rank}(T^k) = \text{rank}(T^{k+m})$ without going through induction, although the proof presented here also gives you an upper bound on k: $k \leq \dim(V)$.

In both parts, the finite-dimension assumption is critical.

2.4.14. Let $V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{F} \right\}$. Construct an isomorphism from V to $\mathbb{F}^3$.

Solution: We will first construct a basis of V before constructing an isomorphism.

Let $\beta = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. It is easy to see that $\beta \subseteq V$ and so $\text{Span}(\beta) \subseteq V$.

Let $A \in V$. Then $A = \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Span}(\beta)$ for some $a, b, c \in \mathbb{F}$. As A is arbitrary, this implies that β spans V.

Let $a, b, c \in \mathbb{F}$ be such that $a \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$. Then $\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so $a = a + b = c = 0$, $b = 0$. As a, b, c are arbitrary, β is linearly independent.

So β is a basis of V, and $\dim(V) = |\beta| = 3 = \dim(\mathbb{F}^3)$.

Let $\Phi : V \to \mathbb{F}^3$ be the linear map that maps β to the standard basis of $\mathbb{F}^3$. By theorem, such Φ exists. By dimension theorem, to show that Φ is an isomorphism, it suffices to show that Φ is onto.

Let $(a, b, c) \in \mathbb{F}^3$. Then $A = \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in V$ and $\Phi(A) = (a, b, c)$ by definition. As (a, b, c) is arbitrary, Φ is onto.

So Φ is an isomorphism from V to $\mathbb{F}^3$.

Note

See also the next question (Question 2.4.15).

2.4.15. Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Solution:

(a) Suppose T is an isomorphism. Then T is one-to-one and onto. By the result of Question 2.1.14 in Homework 2, $T(\beta)$ is a basis for W.

(b) Suppose $T(\beta)$ is a basis for W. Since V and W are both n-dimensional, we may assume that $\beta = \{ v_1, \ldots, v_n \}$ and so $T(\beta) = \{ T(v_1), \ldots, T(v_n) \}$.

- Let $v \in \mathsf{N}(T)$. Then there exist scalars $a_1, \ldots, a_n$ such that $v = \sum_{i=1}^n a_i v_i$. So $0 = T(v) = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i T(v_i)$. Since $T(\beta)$ is a basis for W, it is linearly independent, so $a_1 = \ldots = a_n = 0$ and thus $v = \sum_{i=1}^n a_i v_i = 0$. This implies that T is one-to-one.
- Let $w \in W$. Then there exist scalars $a_1, \ldots, a_n$ such that $w = \sum_{i=1}^n a_i T(v_i) = T(\sum_{i=1}^n a_i v_i) \in \mathsf{R}(T)$. As w is arbitrary, this implies that T is onto.

So T is a bijective linear transformation, thus T is an isomorphism.

Therefore T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Note

By dimension theorem, it suffices to show either condition in part (b). See also the previous question (Question 2.4.14).

2.4.16. Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(\mathbb{F}) \to M_{n\times n}(\mathbb{F})$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

Solution: We first show that Φ is linear: for all $A, A' \in M_{n\times n}(\mathbb{F})$ and $c \in \mathbb{F}$,

- $\Phi(A + A') = B^{-1}(A + A')B = B^{-1}AB + B^{-1}A'B = \Phi(A) + \Phi(A')$
- $\Phi(cA) = B^{-1}(cA)B = cB^{-1}AB = c\Phi(A)$

These imply that Φ is linear.

- Let $A \in \mathsf{N}(\Phi)$. Then $0 = B^{-1}AB$, so $A = B(B^{-1}AB)B^{-1} = B0B^{-1} = 0$. As A is arbitrary, Φ is one-to-one.
- Let $A \in M_{n\times n}(\mathbb{F})$. Then $BAB^{-1} \in M_{n\times n}(\mathbb{F})$, and $\Phi(BAB^{-1}) = B^{-1}(BAB^{-1})B = (B^{-1}B)A(B^{-1}B) = A \in \mathsf{R}(\Phi)$. Since A is arbitrary, Φ is onto.

Thus Φ is an isomorphism.

Note

Since Φ maps a finite-dimensional space $M_{n\times n}(\mathbb{F})$ to itself, by dimension theorem it suffices to show its linearity and either one of one-to-one and onto.

2.4.19. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear mapping defined by $T(M) = M^{\mathsf{T}}$ for each $M \in M_{2\times 2}(\mathbb{R})$.

Let $\beta = \left\{ E^{11}, E^{12}, E^{21}, E^{22} \right\}$, which is a basis of $M_{2\times 2}(\mathbb{R})$.

(a) Compute $[T]_\beta$

(b) Verify that $\mathrm{L}_A \phi_\beta(M) = \phi_\beta T(M)$ for $A = [T]_\beta$ and $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution:

(a) $T(E^{11}) = (E^{11})^\mathsf{T} = 1 \cdot E^{11}$, $T(E^{12}) = 1 \cdot E^{21}$, $T(E^{21}) = 1 \cdot E^{12}$, $T(E^{22}) = 1 \cdot E^{22}$. So $[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot E^{11} + 2 \cdot E^{12} + 3 \cdot E^{21} + 4 \cdot E^{22}$, so $\phi_\beta(M) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and thus $\mathrm{L}_A \phi_\beta(M) = [T]_\beta \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$.

$T(M) = M^\mathsf{T} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = 1 \cdot E^{11} + 3 \cdot E^{12} + 2 \cdot E^{21} + 4 \cdot E^{22}$, so $\phi_\beta T(M) = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$.

Thus the results are consistent.

2.5.3(e). For the following pair of ordered bases β and β' for $\mathsf{P}_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.
$\beta = \{ x^2 - x, x^2 + 1, x - 1 \}$ and $\beta' = \{ 5x^2 - 2x - 3, -2x^2 + 5x + 5, 2x^2 - x - 3 \}$

Idea: The change or coordinate matrix from α-coordinate to β-coordinate is simply $[\mathrm{Id}]_\alpha^\beta$, so to compute the matrix it suffices to simply represent vectors in α in terms of vectors in β.

Solution:

$$\begin{aligned} 5x^2 - 2x - 3 &= 5 \cdot (x^2 - x) + 0 \cdot (x^2 + 1) + 3 \cdot (x - 1) \\ -2x^2 + 5x + 5 &= -6 \cdot (x^2 - x) + 4 \cdot (x^2 + 1) - 1 \cdot (x - 1) \\ 2x^2 - x - 3 &= 3 \cdot (x^2 - x) - 1 \cdot (x^2 + 1) + 2 \cdot (x - 1) \end{aligned}$$

Hence the change of coordinate matrix is $[\mathrm{Id}]_{\beta'}^\beta = \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}$

2.5.4. Let T be the linear operator on $\mathbb{R}^2$ defined by $T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a - 3b \end{pmatrix}$. Let β be the standard ordered basis for $\mathbb{R}^2$, and let $\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$. Use Theorem 2.23 and the fact that $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ to find $[T]_{\beta'}$.

Solution:

$$\begin{aligned} T \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ T \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

So $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$.

$$\begin{aligned} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

So $[\mathrm{Id}]_{\beta'}^\beta = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Hence $[T]_{\beta'} = [\mathrm{Id}]_\beta^{\beta'} [T]_\beta [\mathrm{Id}]_{\beta'}^\beta = \left([\mathrm{Id}]_{\beta'}^\beta\right)^{-1} [T]_\beta [\mathrm{Id}]_{\beta'}^\beta = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}$.

Note

Although $[T]_\beta$ and $[\mathrm{Id}]_{\beta'}^{\beta}$ are easy to compute (as everything is already represented in the standard basis β), usually $\left([\mathrm{Id}]_{\beta'}^{\beta}\right)^{-1}$ and the multiplication of the matrices are much harder and prone to error, especially in the case of high(er) dimensional spaces.

2.5.6(c). For matrix A and ordered basis β, find $[\mathrm{L}_A]_\beta$. Also, find an invertible matrix Q such that $[\mathrm{L}_A]_\beta = Q^{-1}AQ$.

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } \beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

Solution:

$$\mathrm{L}_A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$
$$\mathrm{L}_A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 3 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$
$$\mathrm{L}_A \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 4 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

So $[\mathrm{L}_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$

Let α be the standard basis of $\mathbb{F}^3$. Then by theorem we have $[\mathrm{L}_A]_\alpha = A$, and so $[\mathrm{L}_A]_\beta = [\mathrm{Id}]_\alpha^\beta [\mathrm{L}_A]_\alpha [\mathrm{Id}]_\beta^\alpha = ([\mathrm{Id}]_\beta^\alpha)^{-1} [\mathrm{L}_A]_\alpha [\mathrm{Id}]_\beta^\alpha$, which means we may choose $Q = [\mathrm{Id}]_\beta^\alpha = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

2.5.7. In $\mathbb{R}^2$, let L be the line $y = ma$, where $m \neq 0$. Find an expression for $T(x, y)$, where

(a) T is the reflection of $\mathbb{R}^2$ about L.

(b) T is the projection on L along the line perpendicular to L

Solution:

(a) It is easy to see that T is linear, $T\begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix}$ and $T\begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} m \\ -1 \end{pmatrix}$. Let $\beta = \left\{ \begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ 1 \end{pmatrix} \right\}$. It is easy to verify that β is a basis of $\mathbb{R}^2$. Since T is linear, $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence in the standard ordered basis $\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, $[T]_\alpha = [\mathrm{Id}]_\beta^\alpha [T]_\beta [\mathrm{Id}]_\alpha^\beta = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$, so $T(x, y) = \left(\frac{1-m^2}{1+m^2}x + \frac{2m}{1+m^2}y, \frac{2m}{1+m^2}x - \frac{1-m^2}{1+m^2}y \right)$

(b) It is easy to see that T is linear, $T\begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix}$ and $T\begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let β be defined as in the previous part. As T is linear, $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Hence in the standard ordered basis α, $[T]_\alpha = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}$, so $T(x, y) = \left(\frac{1}{1+m^2}x + \frac{m}{1+m^2}y, \frac{m}{1+m^2}x + \frac{m^2}{1+m^2}y \right)$

Alternatively,

(a) By high school mathematics, the reflected point (a, b) should satisfy that $\left(\frac{a+x}{2}, \frac{b+y}{2}\right) \in L$ and $(a - x, b - y) \in L^\perp$ with $L^\perp$ being the line that passes through origin and is perpendicular to L, hence with slope $-1/m$. This implies that

$\frac{b+y}{2} = m\frac{a+x}{2}$ and $b - y = -\frac{1}{m}(a - x)$. Solving the system gives $(a, b) = \left(\frac{(1-m^2)x+2my}{1+m^2}, \frac{2mx-(1-m^2)y}{1+m^2}\right)$, so $T(x, y) = \left(\frac{1-m^2}{1+m^2}x + \frac{2m}{1+m^2}y, \frac{2m}{1+m^2}x - \frac{1-m^2}{1+m^2}y\right)$

(b) By high school mathematics, the projected point (a, b) should satisfy $(a, b) \in L$ and $(x - y) - (a, b) \in L^\perp$. This implies that $b = ma$ and $y - b = -\frac{1}{m}(x - a)$. Solving the system gives $(a, b) = \left(\frac{x+my}{1+m^2}, \frac{mx+m^2y}{1+m^2}\right)$. So $T(x, y) = \left(\frac{1}{1+m^2}x + \frac{m}{1+m^2}y, \frac{m}{1+m^2}x + \frac{m^2}{1+m^2}y\right)$

Practice Problems

2.2.1. Label the following statements as true or false. Assume that V and W are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $T, U : V \to W$ are linear transformations.

(a) For any scalar a, $aT + U$ is a linear transformation from V to W.

(b) $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies that $T = U$.

(c) If $m = \dim(V)$ and $n = \dim(W)$, then $[T]_\beta^\gamma$ is an $m \times n$ matrix.

(d) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

(e) $\mathcal{L}(V, W)$ is a vector space.

(f) $\mathcal{L}(V, W) = \mathcal{L}(W, V)$.

Solution:

(a) True

(b) True

(c) False. It should be $n \times m$

(d) True

(e) True

(f) False unless $V = W$. Also, they are isomorphic (as they have the same finite dimension)

2.2.2. Let β and γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$

(c) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (2a_2 + a_3, -a_1 + 4a_2 + 5a_3, a_1 + a_3)$

(e) $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(a_1, a_2, \ldots, a_n) = (a_1, a_1, \ldots, a_1)$

(f) $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(a_1, a_2, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$

(g) $T : \mathbb{R}^n \to \mathbb{R}$ defined by $T(a_1, a_2, \ldots, a_n) = a_1 + a_n$

Solution: For the sake of brevity, we will not give the detail proofs for the reasoning. Readers are encouraged to work out the details.

(a) $\begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & \dots & 0 & 1 \end{pmatrix}$

2.2.10. Let V be a vector space with the ordered basis $\beta = \{ v_1, v_2, \dots, v_n \}$. Define $v_0 = 0$. By Theorem 2.6, there exists a linear transformation $T : V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \dots, n$. Compute $[T]_\beta$.

Solution: $T(v_1) = v_0 + v_1 = 1 \cdot v_1$ and $T(v_i) = 1 \cdot v_{i-1} + 1 \cdot v_i$ for all $i \in \{ 2, \dots, n \}$. Hence $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$

2.2.16. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

Idea: To show the proposition, we may try to construct the bases β and γ that satisfy the conditions. Since $[T]_\beta^\gamma$ is diagonal, we must have $T(\beta_i) = c_i \gamma_i$ for some scalar $c_i \in \mathbb{F}$, so it suffices to construct β only. Note that by Question 1.6.34(a) from Homework 2 we should always have $V = \mathsf{N}(T) \oplus U$ where U is some subspace of V that should "correspond to the nonzero part of T". We can then construct β by extending the bases of these subspaces.

Solution: We may assume that $\dim(V) = \dim(W) = n$.

Let $\beta' \subseteq \mathsf{N}(T)$ be a basis of $\mathsf{N}(T) \subseteq V$. By extension theorem, we can extend β' to a basis $\beta \supseteq \beta'$ as $\beta = \beta' \cup \alpha$ where $\alpha \subseteq V$ and $\alpha \cap \beta = \emptyset$. Let $\gamma' = T(\alpha) \subseteq W$.

We may assume that nullity $T = \dim(\mathsf{N}(T)) = k \le n$, and $\alpha = \{ v_1, \dots, v_{n-k} \}$, $\beta' = \{ v_{n-k+1}, \dots, v_n \}$. Then $\gamma' = \{ w_1, \dots, w_{n-k} \}$ with $w_i = T(v_i)$ for $i \in \{ 1, \dots, n-k \}$. We will first show that γ' is linearly independent.

Let $c_1, \dots, c_{n-k} \in \mathbb{F}$ be such that $0 = \sum_{i=1}^{n-k} c_i w_i = \sum_{i=1}^{n-k} c_i T(v_i)$. Then $T(\sum_{i=1}^{n-k} c_i v_i) = 0$, so $\sum_{i=1}^{n-k} c_i v_i \in \mathsf{N}(T) = \mathrm{Span}(\beta')$ and thus there exist $d_{n-k+1}, \dots, d_n \in \mathbb{F}$ such that $\sum_{i=1}^{n-k} c_i v_i = \sum_{i=n-k+1}^{n} d_i v_i$, or $\sum_{i=1}^{n} c_i v_i = 0$ with $c_i = -d_i$ for $i \in \{ n-k+1, \dots, n \}$. Since β is linearly independent, $c_1 = \ldots = c_n = 0$. As $c_1, \dots, c_{n-k}$ are arbitrary, $\gamma' = T(\alpha)$ is linearly independent. In particular, they are distinct.

So we may extend γ' to a basis γ of W as $\gamma = \gamma' \cup \delta$ where $\delta \subseteq W$ and $\gamma' \cap \delta = \emptyset$. We now show that β and γ satisfy the requirement.

For each $i \in \{ 1, \dots, n-k \}$, we have by definition $T(v_i) = w_i$. For each $i \in \{ n-k+1, \dots, n \}$, we have $v_i \in \mathsf{N}(T)$ and so $T(v_i) = 0$. So $[T]_\beta^\gamma = \begin{pmatrix} I_{n-k} & 0_{(n-k)\times k} \\ 0_{k\times(n-k)} & 0_{k\times k} \end{pmatrix}$, which is a diagonal matrix.

Note

Although not presented here, the same proof also works for the case where V, W are not finite dimensional. However, certain concepts (e.g. diagonal matrix) and lemmas (e.g. extension theorem) require extensions first.

2.3.1. Label the following statements as true or false. In each part, V, W, and Z denote vector spaces with ordered (finite) bases α, β, and γ, respectively; $T : V \to W$ and $U : W \to Z$ denote linear transformations; and A and B denote matrices.

(a) $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$
(b) $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ for all $v \in V$
(c) $[U(w)]_\beta = [U]_\alpha^\beta [w]_\beta$ for all $w \in W$
(d) $[\mathrm{Id}_V]_\alpha = I$
(e) $[T^2]_\alpha^\beta = ([T]_\alpha^\beta)^2$
(f) $A^2 = I$ implies that $A = I$ or $A = -I$
(g) $T = \mathrm{L}_A$ for some matrix A
(h) $\mathrm{L}_{A+B} = \mathrm{L}_A + \mathrm{L}_B$
(i) If A is a square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$

Solution:

(a) False unless $V = W = Z$ and $\alpha = \beta = \gamma$, otherwise the multiplication on the right-hand side may not even make sense. Notice the mismatch in the bases

(b) True

(c) False

(d) False unless $V = W$ and $\alpha = \beta$

(e) True

(f) False

(g) False. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(h) True

(i) True

(j) True

2.3.4. For each of the following parts, let T be the linear transformation defined in the corresponding part of Exercise 5 of Section 2.2. Use Theorem 2.14 to compute the following vectors:

(a) $[T(A)]_\alpha$, where $A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$
(b) $[T(f)]_\alpha$, where $f(x) = 4 - 6x + 3x^2$
(c) $[T(A)]_\gamma$, where $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$
(d) $[T(f)]_\gamma$, where $f(x) = 6 - x + 2x^2$

Solution: Recall that

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
$$\gamma = \{1\}$$

(a) Recall that $T(A) = A^\mathsf{T}$. Then $T(A) = A^\mathsf{T} = \begin{pmatrix} 1 & -1 \\ 4 & 6 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 6 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so $[T(A)]_\alpha = \begin{pmatrix} 1 & -1 & 4 & 6 \end{pmatrix}^\mathsf{T}$.

(b) Recall that $T(f) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$. Then $T(f) = \begin{pmatrix} -6 & 2 \\ 0 & 6 \end{pmatrix} = -6 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 6 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so $[T(f)]_\alpha = \begin{pmatrix} -6 & 2 & 0 & 6 \end{pmatrix}^\mathsf{T}$

(c) Recall that $T(A) = \mathrm{tr}(A)$. Then $T(A) = 5 \cdot 1$, so $[T(A)]_\gamma = \begin{pmatrix} 5 \end{pmatrix}$

(d) Recall that $T(f) = f(2)$. Then $T(f) = 12 \cdot 1$, so $[T(f)]_\gamma = \begin{pmatrix} 12 \end{pmatrix}$

2.3.9. Find linear transformations $U, T : \mathbb{F}^2 \to \mathbb{F}^2$ such that $UT = T_0$ and $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = 0$ but $BA \neq 0$.

Solution: Define $T, U : \mathbb{F}^2 \to \mathbb{F}^2$ by $T(a, b) = (a + b, 0)$ and $U(a, b) = (b, 0)$ for $(a, b) \in \mathbb{F}^2$. It is easy to verify that T, U are linear transformations.

For all $(a, b) \in \mathbb{F}^2$ we have $UT(a, b) = U(a + b, 0) = (0, 0)$, so $UT = T_0$. Also, $TU(0, 1) = T(1, 0) = (1, 0) \neq 0_{\mathbb{F}^2}$, so $TU \neq T_0$.

Let α be the standard basis of $\mathbb{F}^2$, and let $A = [U]_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = [T]_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. By the previous result, $AB = [UT]_\alpha = [T_0]_\alpha = 0$ but $BA = [TU]_\alpha \neq [T_0]_\alpha = 0$.

2.3.11. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $\mathsf{R}(T) \subseteq \mathsf{N}(T)$.

Solution:

(a) Suppose $T^2 = T_0$. Let $v \in \mathsf{R}(T)$. Then there exists $w \in V$ such that $T(w) = v$. Then $T(v) = T^2(w) = T_0(w) = 0$, so $v \in \mathsf{N}(T)$. As v is arbitrary, this implies that $\mathsf{R}(T) \subseteq \mathsf{N}(T)$.

(b) Suppose $\mathsf{R}(T) \subseteq \mathsf{N}(T)$. Let $v \in V$. Then $T(v) \in \mathsf{R}(T) \subseteq \mathsf{N}(T)$, so $T^2(v) = T(T(v)) = 0$. As v is arbitrary, this implies that $T^2 = T_0$.

Hence $T^2 = T_0$ if and only if $\mathsf{R}(T) \subseteq \mathsf{N}(T)$.

Note

See also Question 2.1.18 in Homework 2.

2.3.17. Let V be a vector space. Determine all linear transformations $T : V \to V$ such that $T = T^2$.

Idea: It is tempting to say that $T(\mathrm{Id} - T) = 0$ and so either $T = 0$ or $T = \mathrm{Id}$. However, the first equation only can give us that $\mathsf{R}(\mathrm{Id} - T) \subseteq \mathsf{N}(T)$, by the same proof of the previous question (Question 2.3.11).

Assume that V is finite dimensional for the moment. Since $T = T^2$, we have $\mathrm{rank}(T) = \mathrm{rank}(T^2)$. So at least for the case of finite-dimensional V, we have $V = \mathsf{R}(T) \oplus \mathsf{N}(T)$ by Question 2.3.16. It turns out that this condition is exactly what classify such linear maps.

Solution:

(a) Suppose $T : V \to V$ is linear such that $T^2 = T$. Then $\mathsf{R}(T^2) = \mathsf{R}(T)$ and $\mathsf{N}(T) = \mathsf{N}(T^2)$.

Trivially $V \supseteq \mathsf{N}(T) + \mathsf{R}(T)$. Let $v \in V$. Then $v = (v - T(v)) + T(v)$ with $T(v - T(v)) = T(v) - T^2(v) = 0$. This implies that $v - T(v) \in \mathsf{N}(T)$ and $T(v) \in \mathsf{R}(T)$. Hence $v \in \mathsf{N}(T) + \mathsf{R}(T)$. As v is arbitrary, this implies that $V = \mathsf{N}(T) + \mathsf{R}(T)$.

Let $v \in \mathsf{N}(T) \cap \mathsf{R}(T)$. Then there exists $w \in V$ such that $v = T(w)$ and $T(v) = 0$. So $v = T(w) = T^2(w) = T(T(v)) = T(0) = 0$. As v is arbitrary, this implies that $\mathsf{N}(T) \cap \mathsf{R}(T) = \{0\}$.

Thus $V = \mathsf{N}(T) \oplus \mathsf{R}(T)$ is a direct sum decomposition of V.

(b) Let $W_1, W_2 \subseteq V$ be subspaces of V such that $V = W_1 \oplus W_2$. Let β_1, β_2 be bases of W_1, W_2 respectively. By the same proof of Question 2.1.27(a) in Homework 2, the projection map $T : V \to V$ on W_2 along W_1 is well-defined and linear, and by the result of Question 2.1.26 also in Homework 2, we have $\mathsf{R}(T) = W_2 = \{ v \in V : T(v) = v \}$. We now show that $T^2 = T$.

Let $v \in V$. Then $T(v) \in \mathsf{R}(T) = \{ x \in V : T(x) = x \}$, so $T^2(v) = T(T(v)) = T(v)$. As v is arbitrary, $T = T^2$.

So linear transformations $T : V \to V$ such that $T = T^2$ are exactly the projections according to some direct sum decomposition of V.

Note

In Question 2.3.16, we rely on the dimension theorem, and thus the finite-dimension assumption, to obtain that $\mathsf{N}(T) = \mathsf{N}(T^2)$. Here we simply have $T = T^2$, which is a much stronger assumption and directly gives $\mathsf{N}(T) = \mathsf{N}(T^2)$.

You can verify that the constructions done here are inverse to each other and give a bijection between such linear maps and pairs of subspaces as a direct sum decomposition.

It turns out that "every vector space has a basis" and "for every subspace U of a vector space V there exists a subspace $W \subseteq V$ such that $V = U \oplus W$" are equivalent, see this and this answer on MSE.

2.4.1. Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β, respectively, $T : V \to W$ is linear, and A and B are matrices.

(a) $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$
(b) T is invertible if and only if T is one-to-one and onto.
(c) $T = \mathrm{L}_A$, where $A = [T]_\alpha^\beta$
(d) $M_{2\times 3}(\mathbb{F})$ is isomorphic to $\mathbb{F}^5$.
(e) $\mathsf{P}_n(\mathbb{F})$ is isomorphic to $\mathsf{P}_m(\mathbb{F})$ if and only if $n = m$.
(f) $AB = I$ implies that A and B are invertible.
(g) If A is invertible, then $(A^{-1})^{-1} = A$.
(h) A is invertible if and only if L_A is invertible.
(i) A must be square in order to possess an inverse.

Solution:

(a) False unless $V = W$ and $\alpha = \beta$, otherwise it should be $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\beta^\alpha$

(b) True

(c) False unless $V = \mathbb{F}^{|\alpha|}$ and $W = \mathbb{F}^{|\beta|}$, and α, β are the standard bases.

(d) False

(e) True

(f) False. Consider $A = \begin{pmatrix} 0 & 1 \end{pmatrix} \in M_{1\times 2}(\mathbb{F})$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in M_{2\times 1}(\mathbb{F})$.

(g) True

(h) True

(i) True

2.4.2. For each of the following linear transformations T, determine whether T is invertible and justify your answer.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$
(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$
(c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$
(d) $T : \mathsf{P}_3(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ defined by $T(p) = p'$
(e) $T : M_{2\times 2}(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c + d)x^2$
(f) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & a \\ c & c + d \end{pmatrix}$

Solution: For the sake of brevity, we will not give the detail proofs for the reasoning. Readers are encouraged to work out the details.

(a) T is not invertible as it is a mapping between spaces of different dimensions

(b) T is not invertible as it is a mapping between spaces of different dimensions

(c) T is invertible

(d) T is not invertible as it is a mapping between spaces of different dimensions

(e) T is not invertible as it is a mapping between spaces of different dimensions

(f) T is invertible

Note

Although all non-invertible mappings in this question can be noted by comparing the dimension their domains and codomains, not every linear map that maps between spaces of the same dimension is invertible. For example, consider the following map modified from the one in part (f): $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & a \\ c & c+d \end{pmatrix}$

2.4.22. Let $c_0, c_1, \ldots, c_n$ be distinct scalars from an infinite field $\mathbb{F}$. Define $T : \mathsf{P}_n(\mathbb{F}) \to \mathbb{F}^{n+1}$ by $T(f) = (f(c_0), f(c_1), \ldots, f(c_n))$. Prove that T is an isomorphism.

Solution: We first show that T is linear: for $p, q \in \mathsf{P}_n(\mathbb{F})$ and $c \in \mathbb{F}$,

- $T(p+q) = ((p+q)(c_0), (p+q)(c_1), \ldots, (p+q)(c_n)) = (p(c_0), \ldots, p(c_n)) + (q(c_0), \ldots, q(c_n)) = T(p) + T(q)$
- $T(cp) = ((cp)(c_0), \ldots, (cp)(c_n)) = c(p(c_0), \ldots, p(c_n)) = cT(p)$

As p, q, c are arbitrary, T is linear.

Let $p \in \mathsf{N}(\,T\,)$. Then $(p(c_0), \ldots, p(c_n)) = 0_{\mathbb{F}^{n+1}} = (0, \ldots, 0)$, so $p(c_i) = 0$ for all $i \in \{\,0, \ldots, n\,\}$. As $c_0, \ldots, c_n$ are distinct, by factor theorem $p(x) = (x - c_0) \ldots (x - c_n) q(x)$ for some $q \in \mathsf{P}(\mathbb{F})$. Since $p \in \mathsf{P}_n(\mathbb{F})$, $\deg p \le n$, so $\deg q = \deg p - (n+1) \le -1$. This implies that $q = 0$ is the zero polynomial, and so $p = 0$ is the zero polynomial. As p is arbitrary, T is one-to-one.

By dimension theorem, $\operatorname{rank} T = \dim(\mathsf{P}_n(\mathbb{F})) = n + 1 = \dim \mathbb{F}^{n+1}$, so T is onto. This implies that T is an isomorphism.

Note

The requirement that $\mathbb{F}$ is infinite is unnecessary, and the proposition still holds as long as $\mathbb{F}$ has at least $n+1$ elements (so that there are distinct $c_0, \ldots, c_n \in \mathbb{F}$).

See also Shamir's secret sharing scheme.

2.4.23. Let V denote the vector space of sequences in $\mathbb{F}$ that have only a finite number of nonzero terms, and let $W = \mathsf{P}(\mathbb{F})$. Define

$$T : V \to W \quad \text{by} \quad T(\sigma) = \sum_{i=0}^{n} \sigma(i) x^i$$

where n is the largest integer such that $\sigma(n) \neq 0$. Prove that T is an isomorphism.

Solution: We first show that T is linear.

Let $\sigma, \tau \in V$, $c \in \mathbb{F}$. By definition of V, there exist integers $n_\sigma, n_\tau \in \mathbb{N}$ such that $\sigma(n) = 0$ for $n \ge n_\sigma$ and $\tau(n) = 0$ for $n \ge n_\tau$. In particular, there exists $N = \max(n_\sigma, n_\tau)$ such that $\sigma(n) = \tau(n) = 0$ for $n \ge N$, and so $(\sigma + \tau)(n) = 0$ for $n \ge N$. By definition, $T(\sigma) = \sum_{i=0}^{N} \sigma(i) x^i$ and $T(\tau) = \sum_{i=0}^{N} \tau(i) x^i$. Also, $c\sigma(n) = 0$ for all $n \ge N$. Then

- $T(\sigma + \tau) = \sum_{i=0}^{N} (\sigma + \tau)(i) x^i = \sum_{i=0}^{N} = \sum_{i=0}^{N} \sigma(i) x^i + \sum_{i=0}^{N} \tau(i) x^i = T(\sigma) + T(\tau)$
- $T(c\sigma) = \sum_{i=0}^{N} (c\sigma)(i) x^i = c \sum_{i=0}^{N} \sigma(i) x^i = cT(\sigma)$

As σ, τ, c are arbitrary, T is linear.

Let $\sigma \in V$ be a nonzero sequence. Then there exists $n \in \mathbb{N}$ such that $\sigma(n) \neq 0$. By definition of V, there are only finitely many such indices, so we may assume that n is the largest one. Then $T(\sigma) = \sum_{i=0}^{n} \sigma(i) x^i$ with $\sigma(n) \neq 0$. This implies that $T(\sigma) \neq 0$. As σ is arbitrary, this implies that T is one-to-one.

Let $p \in W = \mathsf{P}(\mathbb{F})$. By definition, there exists $n \in \mathbb{N}$ and $c_0, \ldots, c_n \in \mathbb{F}$ such that $p(x) = \sum_{i=0}^{n} c_i x^i$. Define $\sigma : \mathbb{N} \to \mathbb{F}$ by $\sigma(i) = \begin{cases} c_i & \text{if } i \in \{\,0, \ldots, n\,\} \\ 0 & \text{otherwise} \end{cases}$. Since n is finite, it is easy to see that $\sigma \in V$, and $T(\sigma) = \sum_{i=0}^{n} \sigma(i) x^i = \sum_{i=0}^{n} c_i x^i = p$. As p is arbitrary, T is onto.

Hence T is an automorphism.

Note

To show that T is an isomorphism, unlike previous questions we have to show both one-to-one and onto as we can show that the spaces V and $W = \mathsf{P}(\mathbb{F})$ are not finite-dimensional. (You should try to prove them yourself if you have not yet)

2.4.24. Let $T : V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping

$$\bar{T} : V/\mathsf{N}(T) \to Z \quad \text{by} \quad \bar{T}(v + \mathsf{N}(T)) = T(v)$$

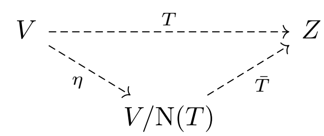
for any coset $v + \mathsf{N}(T)$ in $V/\mathsf{N}(T)$.

(a) Prove that $\bar{T}$ is well-defined; that is, prove that if $v + \mathsf{N}(T) = v' + \mathsf{N}(T)$, then $T(v) = T(v')$

(b) Prove that $\bar{T}$ is linear

(c) Prove that $\bar{T}$ is an isomorphism

(d) Prove that the diagram below commutes; that is, prove that $T = \bar{T}\eta$



Solution:

(a) Let $S \in V/\mathsf{N}(T)$ and $v, v' \in V$ be such that $S = v + \mathsf{N}(T) = v' + \mathsf{N}(T)$. By the result of Question 1.3.31(b) in Homework 1, $v - v' \in \mathsf{N}(T)$, so $T(v - v') = 0$, $T(v) = T(v')$. As S, v, v' is arbitrary, this implies that $\bar{T}$ is well-defined.

(b) Let $S, S' \in V/\mathsf{N}(T)$ and $c \in \mathbb{F}$. By definition, there exist $v, v' \in V$ such that $S = v + \mathsf{N}(T)$ and $S' = v' + \mathsf{N}(T)$. Then

- $\bar{T}(S + S') = \bar{T}((v + \mathsf{N}(T)) + (v' + \mathsf{N}(T))) = \bar{T}((v + v') + \mathsf{N}(T)) = T(v + v') = T(v) + T(v') = \bar{T}(v + \mathsf{N}(T)) + \bar{T}(v' + \mathsf{N}(T)) = \bar{T}(S) + \bar{T}(S')$
- $\bar{T}(cS) = \bar{T}(c(v + \mathsf{N}(T))) = \bar{T}((cv) + \mathsf{N}(T)) = T(cv) = cT(v) = c\bar{T}(v + \mathsf{N}(T)) = c\bar{T}(S)$

As S, S', c are arbitrary, $\bar{T}$ is linear.

(c) Let $S \in \mathsf{N}(\bar{T})$. Then there exists $v \in V$ such that $S = v + \mathsf{N}(T)$. Hence $0 = \bar{T}(S) = \bar{T}(v + \mathsf{N}(T)) = T(v)$ and so $v \in \mathsf{N}(T)$. This implies that $S = v + \mathsf{N}(T) = \mathsf{N}(T) = 0_{V/\mathsf{N}(T)}$. As S is arbitrary, this implies that $\bar{T}$ is one-to-one.

Let $z \in Z$. Since T is onto, there exists $v \in V$ such that $T(v) = z$. Then $v + \mathsf{N}(T) \in V/\mathsf{N}(T)$ and $\bar{T}(v + \mathsf{N}(T)) = T(v) = z$. As z is arbitrary, $\bar{T}$ is onto.

Hence $\bar{T}$ is an automorphism.

(d) Recall that $\eta : V \to V/\mathsf{N}(T)$ is the linear transformation defined by $\eta(v) = v + \mathsf{N}(T)$, as shown in Question 2.1.40 in Homework 2.

Let $v \in V$. Then $\bar{T}\eta(v) = \bar{T}(\eta(v)) = \bar{T}(v + \mathsf{N}(T)) = T(v)$. As v is arbitrary, $\bar{T}\eta = T$.

Note

We can have the same result by replacing Z with $\mathsf{R}(T)$ in case T is not onto.

2.4.25. Let V be a nonzero vector space over a field $\mathbb{F}$, and suppose that S is a basis for V. Let $\mathcal{C}(S, \mathbb{F})$ denote the vector space of all function $f \in \mathcal{F}(S, \mathbb{F})$ such that $f(s) = 0$ for all but a finite number of vectors in S. Let $\Psi : \mathcal{C}(S, \mathbb{F}) \to V$ be defined by $\Psi(f) = 0$ if f is the zero function, and

$$\Psi(f) = \sum_{s \in S,\ f(s) \neq 0} f(s)s$$

otherwise. Prove that Ψ is an isomorphism. Thus every nonzero vector space can be viewed as a space of functions.

Solution: We first show that Ψ is linear.

Let $f, g \in \mathcal{C}(S, \mathbb{F})$, $c \in \mathbb{F}$. Then there exist finite (possibly empty) subsets $S_f, S_g \subseteq S$ such that $f(s) \neq 0$ for $s \in S_f$ and $g(s) \neq 0$ for $s \in S_g$. By definition, $\Psi(f) = \sum_{s \in S_f} f(s)s$ and $\Psi(g) = \sum_{s \in S_g} g(s)s$, with empty set denoting an empty sum and thus the zero vector.

- Let $S_{f+g} = \{ s \in S : (f+g)(s) = f(s) + g(s) \neq 0 \}$. By assumption, $S_{f+g} \subseteq S_f \cup S_g$ where $S_f \cup S_g$ is a finite set, with $(f+g)(s) = 0$ for $s \in (S_f \cup S_g) \setminus S_{f+g}$. So $\Psi(f+g) = \sum_{s \in S_{f+g}} (f+g)(s)s = \sum_{s \in S_f \cup S_g} (f(s) + g(s))s = \sum_{s \in S_f \cup S_g} f(s)s + \sum_{s \in S_f \cup S_g} g(s)s = \sum_{s \in S_f} f(s)s + \sum_{s \in S_g} g(s)s = \Psi(f) + \Psi(g)$
- Let $S_{cf} = \{ s \in S : (cf)(s) = cf(s) \neq 0 \}$. If $c = 0$, we have $S_{cf} = \emptyset \subseteq S_f$. If $c \neq 0$, we have $S_{cf} = \{ s \in S : cf(s) \neq 0 \} = \{ s \in S : f(s) \neq 0 \} = S_f$. Thus $S_{cf} \subseteq S_f$, and $(cf)(s) = 0$ for $s \in S_f \setminus S_{cf}$. Then $\Psi(cf) = \sum_{s \in S_{cf}} (cf)(s)s = \sum_{s \in S_f} cf(s)s = c \sum_{s \in S_f} f(s)s = c\Psi(f)$

As f, g, c are arbitrary, Ψ is linear.

Let $f \in \mathcal{C}(S, \mathbb{F})$ be a nonzero function. Then there exists $s \in S$ such that $f(s) \neq 0$. So $S_f = \{ s \in S : f(s) \neq 0 \}$ is not empty. By assumption, S_f is a finite set, so we may assume that $S_f = \{ s_1, \ldots, s_n \}$ for some $n \in \mathbb{Z}^+$. Then $\Psi(f) = \sum_{s \in S_f} f(s)s = \sum_{i=1}^n f(s_i)s_i$ with all $f(s_1), \ldots, f(s_n)$ being nonzero. Since S is a basis of V, S is linearly independent, and so is $S_f \subseteq S$. This implies that $\Psi(f) \neq 0$. As f is arbitrary, Ψ is one-to-one.

Let $v \in V$. Since S is a basis of V, there exist $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$ be distinct, and $c_1, \ldots, s_n \in \mathbb{F}$ such that $v = \sum_{i=1}^n c_i s_i$. Define $f : S \to \mathbb{F}$ by $f(s) = \begin{cases} c_i & \text{if } s \in \{ s_1, \ldots, s_n \} \\ 0 & \text{otherwise} \end{cases}$. Then by assumption $f \in \mathcal{C}(S, \mathbb{F})$, and $\Psi(f) = \sum_{s \in S,\, f(s) \neq 0} = \sum_{i=1}^n f(s_i)s_i = \sum_{i=1}^n c_i s_i = v$. As v is arbitrary, Ψ is onto.

Therefore Ψ is an isomorphism.

Note

We are always summing over a finite set.

2.5.1. Label the following statements as true or false.

(a) Suppose that $\beta = \{ x_1, x_2, \ldots, x_n \}$ and $\beta' = \{ x_1', x_2', \ldots, x_n' \}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then the jth column of Q is $[x_j]_{\beta'}$.

(b) Every change of coordinate matrix is invertible.

(c) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

(d) The matrices $A, B \in M_{n \times n}(F)$ are called similar if $B = Q^\mathsf{T} AQ$ for some $Q \in M_{n \times n}(F)$.

(e) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[T]_\beta$ is similar to $[T]_\gamma$.

Solution:

(a) False. It should be $[x_j']_\beta$

(b) True

(c) True

(d) False. The definition is $B = Q^{-1}AQ$ for some invertible Q

(e) True

2.5.9. Prove that "is similar to" is an equivalence relation on $M_{n \times n}(\mathbb{F})$.

Solution: Denote the similar relation by $\sim$. Let $A, B, C \in M_{n \times n}(\mathbb{F})$.

- Since $A = (I_n)^{-1} AI_n$, we have $A \sim A$

- Suppose $A \sim B$. Then there exists invertible $Q \in M_{n\times n}(\mathbb{F})$ such that $B = Q^{-1}AQ$. So $A = Q(Q^{-1}AQ)Q^{-1} = \left(Q^{-1}\right)^{-1}BQ^{-1}$ with Q^{-1} being invertible. Hence $B \sim A$
- Suppose $A \sim B$ and $B \sim C$. Then there exist invertible $Q, R \in M_{n\times n}(\mathbb{F})$ such that $B = Q^{-1}AQ$ and $C = R^{-1}BR$. So $C = R^{-1}BR = R^{-1}Q^{-1}AQR = (QR)^{-1}A(QR)$ with QR being invertible. So $A \sim C$

So $\sim$ is reflexive, symmetric and transitive, and hence it is an equivalence relation.

2.5.11. Let V be a finite-dimensional vector space with ordered bases α, β, and γ.

(a) Prove that if Q and R are the change of coordinate matrices that change α-coordinates into β-coordinates and β-coordinates into γ-coordinates, respectively, then RQ is the change of coordinate matrix that changes α-coordinates into γ-coordinates.

(b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

Solution:

(a) By assumption, $Q = [\mathrm{Id}_V]_\alpha^\beta$ and $R = [\mathrm{Id}_V]_\beta^\gamma$. Then $RQ = [\mathrm{Id}_V]_\beta^\gamma[\mathrm{Id}_V]_\alpha^\beta = [\mathrm{Id}_V \circ \mathrm{Id}_V]_\alpha^\gamma = [\mathrm{Id}_V]_\alpha^\beta$, which is the change of coordinate matrix from α-coordinates into γ-coordinates.

(b) By assumption, $Q = [\mathrm{Id}_V]_\alpha^\beta$, so $Q^{-1} = \left([\mathrm{Id}_V]_\alpha^\beta\right)^{-1} = [(\mathrm{Id}_V)^{-1}]_\beta^\alpha = [\mathrm{Id}_V]_\beta^\alpha$, which is the change of coordinate matrix from β-coordinates to α-coordinates.

2.5.13. Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let $\beta = \{ x_1, x_2, \ldots, x_n \}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from $\mathbb{F}$. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij}x_i \quad \text{for} \quad 1 \le j \le n$$

and set $\beta' = \{ x'_1, x'_2, \ldots, x'_n \}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

Solution: Let $c_1, \ldots, c_n \in \mathbb{F}$ be such that $\sum_{j=1}^n c_j x'_j = 0$. Then $0 = \sum_{j=1}^n \sum_{i=1}^n c_j Q_{ij} x_i = \sum_{i=1}^n (Qc)_i x_i$ where $c = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \end{pmatrix}^\mathsf{T} \in \mathbb{F}^n$. Since β is a basis, it is linearly independent, and so $(Qc)_1 = \ldots = (Qc)_n = 0$, which implies that $Qc = 0_{\mathbb{F}^n}$. As Q is invertible, this implies that $c = 0_{\mathbb{F}^n}$, and thus $c_1 = \ldots = c_n = 0$. As $c_1, \ldots, c_n$ are arbitrary, β' is linearly independent. In particular $x'_1, \ldots, x'_n$ are all distinct, and $| \beta' | = | \beta |$.

Since V is finite-dimensional with $\dim V = | \beta | = | \beta' |$ and β' is linearly independent, β' is a basis of V.

Since $x'_j = \sum_{i=1}^n Q_{ij} x_i$, by definition we have $[\mathrm{Id}_V]_\beta^{\beta'} = Q$ is the change of coordinate matrix.