MATH2040 Homework 1
Reference Solution

1.2.13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2)$$

Is V a vector space over $\mathbb{R}$ with those operations? Justify your answer.

Idea: To check if V is a vector space over $\mathbb{R}$, we can check the axioms one by one. However, we can note that the scalar multiplication defined does not affect the second coordinate, which combined with the addition could result in an example that violates one of the axioms.

Solution: V is not a vector space as it does not satisfy distributivity with respect to scalar addition (VS 8): $(0, 2) \in V$ and $1 \in \mathbb{R}$, but $(1 + 1) \cdot (0, 2) = 2 \cdot (0, 2) = (0, 2) \neq (0, 4) = (0, 2) + (0, 2) = 1 \cdot (0, 2) + 1 \cdot (0, 2)$.

Note

There are other axioms that are not satisfied. To show that V is not a vector space, we only need to find one such example.

1.2.18. Let $V = \{ (a_1, a_2) : a_1, a_2 \in \mathbb{R} \}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$ define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2)$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution: V is not a vector space as it does not satisfy commutativity of addition (VS 1): $(0, 0), (0, 1) \in V$, but $(0, 0)+(0, 1) = (0 + 2 \cdot 0, 0 + 3 \cdot 1) = (0, 3) \neq (0, 1) = (0 + 2 \cdot 0, 1 + 3 \cdot 0) = (0, 1) + (0, 0)$.

1.2.19. Let $V = \{ (a_1, a_2) : a_1, a_2 \in \mathbb{R} \}$. Define addition of elements of V coordinatewise, and for (a_1, a_2) in V and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0 \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution: V is not a vector space as it does not satisfy distributivity with respect to scalar addition (VS 8): $(0, 2) \in V$ and $1 \in \mathbb{R}$, but $(1 + 1) \cdot (0, 2) = 2 \cdot (0, 2) = (0, 1) \neq (0, 4) = (0, 2) + (0, 2) = 1 \cdot (0, 2) + 1 \cdot (0, 2)$.

1.3.11. Is the set $W = \{ f(x) \in \mathsf{P}(\mathbb{F}) : f(x) = 0 \text{ or } f(x) \text{ has degree } n \}$ a subspace of $\mathsf{P}(\mathbb{F})$ if $n \geq 1$? Justify your answer.

Idea: To check if W is a subspace of $\mathsf{P}(\mathbb{F})$, we can check the 3 conditions in the definition one by one. However, we can note that the definition for W (on nonzero polynomial) depends only on the degree, which is not closed under addition on polynomial.

Solution: W is not a vector for all possible $n \geq 1$.

Let $f(x) = x^n + 1$ and $g(x) = -x^n$. Then $f(x), g(x)$ both have degree n, so $f(x), g(x) \in W$, but $f(x) + g(x) = (x^n + 1) + (-x^n) = 1$ has degree $0 \neq n$ and is a nonzero polynomial, so $f(x) + g(x) = 1 \notin W$. This implies that W is not closed under addition, and so not a subspace of $\mathsf{P}(\mathbb{F})$.

1.3.19. Let W_1 and W_2 be subspaces of a vectors space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Solution:

(a) Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ holds. By symmetry (on the indices) we may assume that $W_1 \subseteq W_2$ holds. Then $W_1 \cup W_2 = W_2$, which by assumption is a subspace of V.

(b) Suppose neither $W_1 \subseteq W_2$ nor $W_2 \subseteq W_1$ holds. Then there exists $w_1 \in W_1 \setminus W_2$ and $w_2 \in W_2 \setminus W_1$. By definition, $w_1, w_2 \in W_1 \cup W_2$.

If $w_1 + w_2 \in W_1 \cup W_2$, it would be true that $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$. By symmetry, we may assume that $w_1 + w_2 \in W_1$. As W_1 is a subspace, we have $-w_1 \in W_1$ and so $w_2 = (w_1 + w_2) + (-w_1) \in W_1$. Contradiction arises. Hence $w_1 + w_2 \notin W_1 \cup W_2$.

This implies that $W_1 \cup W_2$ is not closed under addition, and so is not a subspace of V.

Hence, $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \cup W_2$ or $W_2 \cup W_1$.

1.3.23. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contains $W_1 + W_2$

Solution:

(a) We first verify that $W_1 + W_2$ is a subspace of V.

- Since W_1, W_2 are subspaces of V, $0 \in W_1$ and $0 \in W_2$, so $0 = 0 + 0 \in W_1 + W_2$.
- Let $x, y \in W_1 + W_2$. By definition, there exists $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ such that $x = x_1 + x_2$ and $y = y_1 + y_2$. As W_1, W_2 are subspaces, $x_1 + y_1 \in W_1$ and $x_2 + y_2 \in W_2$. So $x + y = (x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) \in W_1 + W_2$.
- Let $x \in W_1 + W_2$ and $c \in \mathbb{F}$ be a scalar. Then there exists $x_1 \in W_1$ and $x_2 \in W_2$ such that $x = x_1 + x_2$. Since W_1, W_2 are subspaces, $cx_1 \in W_1$ and $cx_2 \in W_2$. So $cx = c(x_1 + x_2) = (cx_1) + (cx_2) \in W_1 + W_2$.

By definition, $W_1 + W_2$ is a subspace of V.

We now show that $W_1 + W_2$ contains both W_1 and W_2.

Let $w_1 \in W_1$. Since W_2 is a subspace of V, $0 \in W_2$. So $w_1 = w_1 + 0 \in W_1 + W_2$. As w_1 is arbitrary, $W_1 \subseteq W_1 + W_2$.

Let $w_2 \in W_2$. Since W_1 is a subspace of V, $0 \in W_1$. So $w_2 = 0 + w_2 \in W_1 + W_2$. As w_2 is arbitrary, $W_2 \subseteq W_1 + W_2$.

So $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Let $U \subseteq V$ be a subspace of V that contains both W_1 and W_2.

Let $w \in W_1 + W_2$. Then there exists $w_1 \in W_1$ and $w_2 \in W_2$ such that $w = w_1 + w_2$. Since $W_1 \subseteq U$ and $W_2 \subseteq U$, $w_1, w_2 \in U$. Since U is a subspace, $w = w_1 + w_2 \in U$.

As w is arbitrary, $W_1 + W_2 \subseteq U$.

As U is arbitrary, every subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Note

Together with the previous question (Question 1.3.19), we can see that in the scope of linear algebra, set addition (or Minkowski sum) plays the role of joining sub-structures as set union does in set theory, as set union is no longer suitable for this purpose (except for degenerate cases).

See also Question 1.4.14.

1.3.26. In $M_{m\times n}(\mathbb{F})$ define $W_1 = \{ A \in M_{m\times n}(\mathbb{F}) : A_{ij} = 0 \text{ whenever } i > j \}$ and $W_2 = \{ A \in M_{m\times n}(\mathbb{F}) : A_{ij} = 0 \text{ whenever } i \leq j \}$. Show that $M_{m\times n}(\mathbb{F}) = W_1 \oplus W_2$.

Idea: To show that $M_{m\times n}(\mathbb{F}) = W_1 \oplus W_2$, we need justify first that W_1 and W_2 are both subspaces of $M_{m\times n}(\mathbb{F})$ (as it is not yet clear), then show that both $M_{m\times n}(\mathbb{F}) = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$ hold, as required by the definition of direct sum.

Solution: Let $V = M_{n\times m}(\mathbb{F})$.

We first show that W_1, W_2 are subspaces of V.

Let $0_{m\times n}$ denote that zero matrix in $V = M_{m\times n}(\mathbb{F})$.

- For all $i > j$ we have $(0_{m\times n})_{ij} = 0$, so $0_{m\times n} \in W_1$.

  Similarly, for all $i \le j$ we have $(0_{m\times n})_{ij} = 0$, so $0_{m\times n} \in W_2$.
- Let $A, B \in W_1$. Then $A_{ij} = B_{ij} = 0$ for all $i > j$, hence $(A+B)_{ij} = A_{ij} + B_{ij} = 0$ for all $i > j$. This implies that $A + B \in W_1$.

  Similarly, let $A, B \in W_2$. Then $A_{ij} = B_{ij} = 0$ for all $i \le j$, hence $(A+B)_{ij} = A_{ij} + B_{ij} = 0$ for all $i \le j$. This implies that $A + B \in W_2$.
- Let $A \in W_1$, $c \in \mathbb{F}$. Then $A_{ij} = 0$ for all $i > j$, so $(cA)_{ij} = cA_{ij} = 0$ for all $i > j$. This implies that $cA \in W_1$.

  Similarly, let $A \in W_2$, $c \in \mathbb{F}$. Then $A_{ij} = 0$ for all $i \le j$, so $(cA)_{ij} = cA_{ij} = 0$ for all $i \le j$. This implies that $cA \in W_2$.

This implies that W_1, W_2 are subspaces of V.

To show that $V = W_1 \oplus W_2$, we first show that $V = W_1 + W_2$.

Since $W_1, W_2 \subseteq V$, $w_1 + w_2 \in V$ for all $w_1 \in W_1$, $w_2 \in W_2$. Hence $W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1, w_2 \in W_2 \} \subseteq V$.

Let $A \in V$. Define $A_1, A_2 \in V$ as

$$(A_1)_{ij} = \begin{cases} A_{ij} & \text{if } i \le j \\ 0 & \text{if } i > j \end{cases} \quad \text{and} \quad (A_2)_{ij} = \begin{cases} 0 & \text{if } i \le j \\ A_{ij} & \text{if } i > j \end{cases} \quad \text{for all} \quad i \in \{ 1, \dots, m \}, j \in \{ 1, \dots, n \}$$

By definition, $A_1 \in W_1$ and $A_2 \in W_2$. Also, for all $i \in \{ 1, \dots, m \}$ and $j \in \{ 1, \dots, n \}$,

$$(A_1 + A_2)_{ij} = \begin{cases} A_{ij} + 0 & \text{if } i \le j \\ 0 + A_{ij} & \text{if } i > j \end{cases} = A_{ij}$$

So $A = A_1 + A_2 \in W_1 + W_2$. As A is arbitrary, $V \subseteq W_1 + W_2$.

Thus $V = W_1 + W_2$.

We now show that $W_1 \cap W_2 = \{0_{m\times n}\}$.

Since W_1, W_2 are subspaces of V, $0_{m\times n} \in W_1$ and $0_{m\times n} \in W_2$. Thus $0_{m\times n} \in W_1 \cap W_2$, $\{0_{m\times n}\} \subseteq W_1 \cap W_2$.

Let $A \in W_1 \cap W_2$. Then $A_{ij} = 0$ for all $i > j$ and $A_{ij} = 0$ for all $i \le j$. So $A_{ij} = 0$ for all $i \in \{ 1, \dots, m \}$, $j \in \{ 1, \dots, n \}$. This implies that $A = 0_{m\times n}$. As A is arbitrary, $W_1 \cap W_2 \subseteq \{0_{m\times n}\}$.

Thus $W_1 \cap W_2 = \{0_{m\times n}\}$

This implies that $M_{m\times n}(\mathbb{F}) = W_1 \oplus W_2$.

1.3.31. Let W be a subspace of a vector space V over a field $\mathbb{F}$.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$

(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$

(c) Prove that the operations on the collection $S = \{ v + W : v \in V \}$ are well-defined; that is, show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$$

and

$$a(v_1 + W) = a(v_1' + W)$$

for all $a \in \mathbb{F}$

(d) Prove that the set S is a vector space with the operations defined.

Solution:

(a) Suppose $v + W$ is a subspaces of V. Then $0 \in v + W$, and so $v + w = 0$ for some $w \in W$. This means that $w = -v \in W$. As W is a subspace of V, $v = -(-v) = -w \in W$.

Suppose $v \in W$. We will show that $v + W = W$, which by assumption is a subspace of V:

- For all $w \in W$ we have $v + w \in W$, so $v + W = \{ v + w : w \in W \} \subseteq W$.
- Let $w \in W$. Then we have $w - v \in W$ and so $w = v + (w - v) \in v + W$. As w is arbitrary, $W \subseteq v + W$.

So $v + W = W$.

(b) Suppose $v_1 + W = v_2 + W$. Since W is a subspace of V, $0 \in W$. So $v_1 = v_1 + 0 \in v_1 + W = v_2 + W$. Hence there exists $w \in W$ such that $v_1 = v_2 + w$, and so $v_1 - v_2 = w \in W$.

Suppose $v_1 - v_2 \in W$. As W is a subspace of V, $v_2 - v_1 = -(v_1 - v_2) \in W$.

- Let $v_1 + w \in v_1 + W$ for some $w \in W$. Then $(v_1 - v_2) + w \in W$, so $v_1 + w = v_2 + (v_1 - v_2) + w \in v_2 + W$. As $v_1 + w$ is arbitrary, $v_1 + W \subseteq v_2 + W$.
- By symmetry (on the indices), we also have $v_2 + W \subseteq v_1 + W$.

This implies that $v_1 + W = v_2 + W$.

(c) Let $v_1, v_1', v_2, v_2' \in V$ be such that $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, and $a \in \mathbb{F}$. By the previous part, $v_1 - v_1' \in W$ and $v_2 - v_2' \in W$.

i. As W is a subspace, we have $(v_1 + v_2) - (v_1' + v_2') = (v_1 - v_1') + (v_2 - v_2') \in W$. By the previous part, this implies that $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W = (v_1' + v_2') + W = (v_1' + W) + (v_2') + W$.

ii. Since W is a subspace, we have $(av_1) - (av_1') = a(v_1 - v_1') \in W$. By the previous part, $a(v_1 + W) = (av_1) + W = (av_1') + W = a(v_1' + W)$.

Hence the operations on S do not depend on the representation of the elements, and so are well-defined.

(d) To show that S, equipped with the operations defined, is a vector space over $\mathbb{F}$, we verify all axioms one by one:

1. Let $(x + W), (y + W) \in S$. Then $(x + W) + (y + W) = (x + y) + W = (y + x) + W = (y + W) + (x + W)$
2. Let $(x + W), (y + W), (z + W) \in S$. Then $((x + W) + (y + W)) + (z + W) = ((x + y) + W) + (z + W) = (x + y + z) + W = (x + W) + ((y + z) + W) = (x + W) + ((y + W) + (z + W))$
3. Denote $\vec{0} = W = 0 + W$. Then for all $(x + W) \in S$, $(x + W) + \vec{0} = (x + W) + (0 + W) = x + W$
4. Let $x + W \in S$ with $x \in V$. Then $(-x) + W \in S$, and $(x + W) + ((-x) + W) = (x + (-x)) + W = 0 + W = \vec{0}$
5. Let $x + W \in S$. Then $1 \cdot (x + W) = (1 \cdot x) + W = x + W$
6. Let $x + W \in S$, $a, b \in \mathbb{F}$. Then $a \cdot (b \cdot (x + W)) = a \cdot ((bx) + W) = (abx) + W = (ab) \cdot (x + W)$
7. Let $a \in \mathbb{F}$ and $(x + W), (y + W) \in S$. Then $a \cdot ((x + W) + (y + W)) = a \cdot ((x + y) + W) = (a \cdot (x + y)) + W = (ax + ay) + W = ((ax) + W) + ((ay) + W) = a \cdot (x + W) + a \cdot (y + W)$
8. Let $a, b \in \mathbb{F}$ and $x + W \in S$. Then $(a + b) \cdot (x + W) = ((a + b) \cdot x) + W = (ax + bx) + W = ((ax) + W) + ((bx) + W) = a \cdot (x + W) + b \cdot (x + W)$

Since all axioms are satisfied, S is a vector space over $\mathbb{F}$ with the operations defined.

1.4.5. In each part, determine whether the given vector is in the span of S.

(a) $(2, -1, 1)$, $S = \{ (1, 0, 2), (-1, 1, 1) \}$

(b) $(-1, 2, 1)$, $S = \{ (1, 0, 2), (-1, 1, 1) \}$

(c) $(-1, 1, 1, 2)$, $S = \{ (1, 0, 1, -1), (0, 1, 1, 1) \}$

(d) $(2, -1, 1, -3)$, $S = \{ (1, 0, 1, -1), (0, 1, 1, 1) \}$

(e) $-x^3 + 2x^2 + 3x + 3$, $S = \left\{ x^3 + x^2 + x + 1, x^2 + x + 1, x + 1 \right\}$

(f) $2x^3 - x^2 + x + 3$, $S = \left\{ x^3 + x^2 + x + 1, x^2 + x + 1, x + 1 \right\}$

(g) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

(h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

Idea: To show that a given vector is spanned by a set, it suffices to express the vector as a linear combination of vectors from the set. On the other hand, to show that a given vector is not spanned by the set, it is necessary to show that no linear combination of vectors from the sets equals to the given vector.

A common way to tackle this kind of problem is to find the coefficients of the correct linear combination. This works when the set S is a finite set. Usually, for subsets of spaces like $\mathbb{R}^n$ and $\mathbb{C}^n$, the coefficients would satisfy a system of linear equations, which can be solved with techniques from MATH1030.

Solution:

(a) $(2,-1,1) \in \text{Span}(\ S\ )$ as $(2,-1,1) = 1\cdot(1,0,2) - 1\cdot(-1,1,1)$

(b) $(-1,2,1) \notin \text{Span}(\ S\ )$. Suppose $(-1,2,1) = a\cdot(1,0,2) + b\cdot(-1,1,1) = (a-b, b, 2a+b)$ for some scalars a, b. Then we must have $a-b=-1$, $b=2$, $2a+b=1$. However, the system does not have any solution. So $(-1,2,1)$ cannot be spanned by S.

(c) $(-1,1,1,2) \notin \text{Span}(\ S\ )$. Suppose $(-1,1,1,2) = a\cdot(1,0,1,-1) + b\cdot(0,1,1,1) = (a, b, a+b, -a+b)$ for some scalars a, b. Then we must have $a=-1$, $b=1$, $a+b=1$, $-a+b=2$. However, the system does not have any solution. So $(-1,1,1,2)$ cannot be spanned by S.

(d) $(2,-1,1,-3) \in \text{Span}(\ S\ )$ as $(2,-1,1,-3) = 2\cdot(1,0,1,-1) - 1\cdot(0,1,1,1)$

(e) $-x^3+2x^2+3x+3 \in \text{Span}(\ S\ )$ as $-x^3+2x^2+3x+3 = -1\cdot(x^3+x^2+x+1) + 3\cdot(x^2+x+1) + 1\cdot(x+1)$

(f) $2x^3-x^2+x+3 \notin \text{Span}(\ S\ )$. Suppose $2x^3-x^2+x+3 = a\cdot(x^3+x^2+x+1) + b\cdot(x^2+x+1) + c\cdot(x+1) = ax^3+(a+b)x^2+(a+b+c)x+(a+b+c)$ for some scalars a, b, c. Then we must have $a=2$, $a+b=-1$, $a+b+c=1$, $a+b+c=3$. However, the system does not have any solution. So $-x^3+2x^2+3x+3$ cannot be spanned by S

(g) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \in \text{Span}(\ S\ )$ as $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = 3\cdot\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + 4\cdot\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - 2\cdot\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \text{Span}(\ S\ )$. Suppose $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a\cdot\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\cdot\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c\cdot\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+c & b+c \\ -a & b \end{pmatrix}$ for some scalars a, b, c. Then we must have $a+c=1$, $b+c=0$, $-a=0$, $b=1$. However, the system does not have any solution. So $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ cannot be spanned by S.

1.4.10. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{\ M_1, M_2, M_3\ \}$ is the set of all symmetric 2×2 matrices.

Solution: Denote the set of 2×2 symmetric matrices by $S \subseteq M_{2\times 2}(\mathbb{F})$.

Let $A \in S \subseteq M_{2\times 2}(\mathbb{F})$. Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $a, b, c, d \in \mathbb{F}$. As A is symmetric, $b = c$. So $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d\cdot\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = a\cdot M_1 + d\cdot M_2 + b\cdot M_3 \in \text{Span}(\ \{\ M_1, M_2, M_3\ \}\ )$. As A is arbitrary, $S \subseteq \text{Span}(\ \{\ M_1, M_2, M_3\ \}\ )$.

Before showing that $\text{Span}(\ \{\ M_1, M_2, M_3\ \}\ ) \subseteq S$, we will first show that S is a subspace of $M_{2\times 2}(\mathbb{F})$. It is easy to verify that S satisfies all conditions:

- The zero matrix $0_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a symmetric 2×2 matrix, and so $0_{2\times 2} \in S$
- Let $A_1, A_2 \in S$. Then $A_1 = \begin{pmatrix} a_1 & b_1 \\ b_1 & d_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} a_2 & b_2 \\ b_2 & d_2 \end{pmatrix}$ for some scalars $a_1, b_1, d_1, a_2, b_2, d_2$. So $A_1 + A_2 = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & d_1+d_2 \end{pmatrix}$, which is a symmetric 2×2 matrix, so $A_1 + A_2 \in S$
- Let $A \in S$, $c \in \mathbb{F}$. Then $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ for some scalars a, b, d. Then $cA = \begin{pmatrix} ca & cb \\ cb & cd \end{pmatrix}$, which is a symmetric 2×2 matrix, so $cA \in S$

Hence S is a subspace of $M_{2\times 2}(\mathbb{F})$.

Since each of M_1, M_2, M_3 is a symmetric 2×2 matrix, $\{\ M_1, M_2, M_3\ \} \subseteq S$, so $\text{Span}(\ \{\ M_1, M_2, M_3\ \}\ ) \subseteq S$ by the property of span.

Therefore, $S = \text{Span}(\ \{\ M_1, M_2, M_3\ \}\ )$.

1.4.14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{Span}(\ S_1 \cup S_2\ ) = \text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ )$.

Solution: By the property of span, we have $S_1 \subseteq \text{Span}(\ S_1\ )$.

Since $\text{Span}(\ S_2\ )$ is a subspace of V, $0 \in \text{Span}(\ S_2\ )$, and so $\text{Span}(\ S_1\ ) = \text{Span}(\ S_1\ ) + \{0\} \subseteq \text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ )$.

Thus, $S_1 \subseteq \text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ )$. Similarly, we have $S_2 \subseteq \text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ )$, so $S_1 \cup S_2 \subseteq \text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ )$.

By Question 1.3.23, $\text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ )$, being a sum of subspaces $\text{Span}(\ S_1\ )$ and $\text{Span}(\ S_2\ )$, is a subspace of V. So by the property of span we have $\text{Span}(\ S_1 \cup S_2\ ) \subseteq \text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ )$.

Trivially, $S_1 \subseteq S_1 \cup S_2 \subseteq \text{Span}(\ S_1 \cup S_2\ )$, so $\text{Span}(\ S_1\ ) \subseteq \text{Span}(\ S_1 \cup S_2\ )$. Similarly, $\text{Span}(\ S_2\ ) \subseteq \text{Span}(\ S_1 \cup S_2\ )$. By Question 1.3.23, $\text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ ) \subseteq \text{Span}(\ S_1 \cup S_2\ )$.

Therefore, $\text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ ) = \text{Span}(\ S_1 \cup S_2\ )$.

Note

Do not assume that S_1 or S_2 is a finite set. In particular, do **not** write something like $S = \{\ v_1, v_2, \ldots, v_n\ \}$ unless you know already that S is a (nonempty) finite set. This is also the reason why the above proof utilizes the properties of span and sum (from Question 1.3.23) instead of going through the definitions (in particular, the definition of span).

With an appropriate definition on sum, this proposition can be generalized beyond summing two spans to summing an arbitrary collection of spans: $\sum_\alpha \text{Span}(\ S_\alpha\ ) = \text{Span}(\ \bigcup_\alpha S_\alpha\ )$. Together with Question 1.4.12, we have a way of joining an arbitrary collection of subspaces $\{U_\alpha\}$: it is just $\text{Span}(\ \bigcup_\alpha U_\alpha\ )$

1.4.15. Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{Span}(\ S_1 \cap S_2\ ) \subseteq \text{Span}(\ S_1\ ) \cap \text{Span}(\ S_2\ )$. Give an example in which $\text{Span}(\ S_1 \cap S_2\ )$ and $\text{Span}(\ S_1\ ) \cap \text{Span}(\ S_2\ )$ are equal and one in which they are unequal.

Solution: Since $S_1 \cap S_2 \subseteq S_1 \subseteq \text{Span}(\ S_1\ )$, we have by the property of span that $\text{Span}(\ S_1 \cap S_2\ ) \subseteq \text{Span}(\ S_1\ )$. Similarly, we have $\text{Span}(\ S_1 \cap S_2\ ) \subseteq \text{Span}(\ S_2\ )$. So $\text{Span}(\ S_1 \cap S_2\ ) \subseteq \text{Span}(\ S_1\ ) \cap \text{Span}(\ S_2\ )$.

To give examples, consider $V = \mathbb{R}$, the (usual) real line:

- For $S_1 = S_2 = \emptyset$, we have $\text{Span}(\ S_1\ ) = \text{Span}(\ S_2\ ) = \text{Span}(\ S_1 \cap S_2\ ) = \{0\}$
- For $S_1 = \{\ 1\ \}$ and $S_2 = \{\ 2\ \}$, we have $\text{Span}(\ S_1\ ) = \text{Span}(\ S_2\ ) = V$ but $\text{Span}(\ S_1 \cap S_2\ ) = \text{Span}(\ \emptyset\ ) = \{0\} \neq V$

Note

The existences of these examples are mostly artifact of the distinction between subsets and subspaces. See also Question 1.4.12.

Practice Problems

1.2.1. Label the following statements as true or false.

(a) Every vector space contains a zero vector.

(b) A vector space may have more than one zero vector.

(c) In any vector space, $ax = bx$ implies that $a = b$.

(d) In any vector space, $ax = ay$ implies that $x = y$.

(e) A vector in $\mathbb{F}^n$ may be regarded as a matrix in $M_{n\times 1}(\mathbb{F})$

(f) An $m \times n$ matrix has m columns and n rows.

(g) In $\mathsf{P}(\mathbb{F})$, only polynomials of the same degree may be added.

(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.

(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.

Solution:

(a) True

(b) False. If $0, 0'$ are both zero vectors of a vector space, we must have $0 = 0 + 0' = 0'$

(c) False. Consider $x = 0$, the zero vector

(d) False. Consider $a = 0$, the zero scalar

(e) True

(f) False. It should be m rows and n columns

(g) False

(h) False. See Question 1.3.11

(i) True

1.2.8. In any vector space V, show that $(a+b)(x+y) = ax + by + bx + ay$ for any $x, y \in V$ and any $a, b \in \mathbb{F}$

Solution: By the axioms of vector space, $(a+b)(x+y) = (a+b)x + (a+b)y = ax + bx + ay + by$

1.2.14. Let $V = \{ (a_1, a_2, \ldots, a_n) : a_i \in \mathbb{C} \text{ for } i = 1, 2, \ldots, n \}$; so V is a vector space over $\mathbb{C}$. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

Solution: $(V, \mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}})$ is a real vector space. All axioms can be verified directly as V itself is a complex vector space, which we will omit the detailed steps here. The only axioms that are relevant are those that involve scalar multiplication, which still hold true due to the fact that $\mathbb{R}$ is a subfield of $\mathbb{C}$.

1.2.15. Let $V = \{ (a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n \}$; so V is a vector space over $\mathbb{R}$. Is V a vector space over the field of complex numbers with the operations of coordinatewise addition and multiplication?

Solution: $(V, \mathbb{C}, +_{\mathbb{C}}, \cdot_{\mathbb{C}})$ is not a complex vector space: $(1, 0, \ldots, 0) \in V$ but $i \cdot (1, \ldots, 0) = (i, 0, \ldots, 0) \notin V$ as $i \notin \mathbb{R}$.

1.2.21. Let V and W be vector spaces over a field $\mathbb{F}$. Let

$$Z = \{ (v, w) : v \in V \text{ and } w \in W \}$$

Prove that Z is a vector space over $\mathbb{F}$ with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cw_1, cw_2)$$

Solution: We verify all axioms one by one:

1. Let $x = (v_x, w_x), y = (v_y, w_y) \in Z$ with $v_x, v_y \in V$ and $w_x, w_y \in W$. Then $x + y = (v_x, w_x) + (v_y, w_y) = (v_x + v_y, w_x + w_y) = (v_y + v_x, w_y + w_x) = (v_y, w_y) + (v_x, w_x) = y + x$
2. Let $x = (v_x, w_x), y = (v_y, w_y), z = (v_z, w_z) \in Z$ with $v_x, v_y, v_z \in V$ and $w_x, w_y, w_z \in W$. Then $(x + y) + z = ((v_x, w_x) + (v_y, w_y)) + (v_z, w_z) = (v_x + v_y, w_x + w_y) + (v_z, w_z) = (v_x + v_y + v_z, w_x + w_y + w_z) = (v_x, w_x) + (v_y + v_z, w_y + w_z) = (v_x, w_x) + ((v_y, w_y) + (v_z, w_x)) = x + (y + z)$
3. Let $\vec{0} = (0_V, 0_W)$ where $0_V, 0_W$ are the zero vectors of V, W respectively. Then $\vec{0} \in Z$ and for all $x = (v_x, w_x) \in Z$, $x + \vec{0} = (v_x, w_x) + (0_V, 0_W) = (v_x + 0_V, w_x + 0_W) = (v_x, w_x) = x$
4. Let $x = (v_x, w_x) \in Z$. Then for $y = (-v_x, -w_x)$, $y \in Z$ and $x + y = (v_x, w_x) + (-v_x, -w_x) = (v_x - v_x, w_x - w_x) = (0_V, 0_W) = \vec{0}$
5. Let $x = (v_x, w_x) \in Z$. Then $1 \cdot x = 1 \cdot (v_x, w_x) = (1 \cdot v_x, 1 \cdot w_x) = (v_x, w_x) = x$

6. Let $x = (v_x, w_x) \in Z$ and $a, b \in \mathbb{F}$. Then $a \cdot (b \cdot x) = a \cdot (b \cdot (v_x, w_x)) = a \cdot (bv_x, bw_x) = (abv_x, abw_x) = (ab) \cdot (v_x, w_x) = (ab) \cdot x$

7. Let $x = (v_x, w_x), y = (v_y, w_y) \in Z$ and $a \in \mathbb{F}$. Then $a \cdot (x + y) = a \cdot ((v_x, w_x) + (v_y, w_y)) = a \cdot (v_x + v_y, w_x + w_y) = (a \cdot (v_x + v_y), a \cdot (w_x + w_y)) = (av_x + av_y, aw_x + aw_y) = (av_x, av_y) + (aw_x, aw_y) = a \cdot (v_x, w_x) + a \cdot (v_y, w_y) = a \cdot x + a \cdot y$

8. Let $x = (v_x, w_x) \in Z$ and $a, b \in \mathbb{F}$. Then $(a+b) \cdot x = (a+b) \cdot (v_x, w_x) = ((a+b) \cdot v_x, (a+b) \cdot w_x) = (av_x + bv_x, aw_x + bw_x) = (av_x, aw_x) + (bv_x, bw_x) = a \cdot (v_x, w_x) + b \cdot (v_x, w_x) = a \cdot x + b \cdot x$

As the axioms are satisfied, Z is a vector space over $\mathbb{F}$ with the operations defined.

Note

You cannot simply claim that Z is a vector space by just checking the 3 conditions in the definition for subspaces. If you want to do so, you will need to first show that Z is contained (as a subset) of some known vector space that would give the same addition and scalar multiplication on Z. In this case a natural choice will (most likely) be Z itself, which we do not (yet) have a vector space structure on Z (in fact, this is exactly what this question asks for).

1.3.1. Label the following statements as true or false.

(a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.

(b) The empty set is a subspace of every vector space.

(c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

(d) The intersection of any two subsets of V is a subspace of V.

(e) An $n \times n$ diagonal matrix can never have more than n nonzero entries.

(f) The trace of a square matrix is the product of its diagonal entries.

(g) Let W be the xy-plane in $\mathbb{R}^3$. Then $W = \mathbb{R}^2$

Solution:

(a) False, unless the vector space structure on W is consistent with the one on V. For example, consider $V = \mathbb{C}^2$ the same as Question 1.2.14, then $W = \{ (x, y) \in V : x, y \in \mathbb{R} \}$ is a (real) vector space with the usual operations on the scalar field $\mathbb{R} \subsetneq \mathbb{C}$ (which gives the structure equivalent to that on $\mathbb{R}^2$), but W is not a subspace of the complex vector space V.

(b) False. The empty set is not a vector space.

(c) True. Consider $W = \{0_V\}$

(d) False

(e) True

(f) False

(g) False. Note that they are only *equivalent / isomorphic*, not *equal*

1.3.8. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

(a) $W_1 = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2 \right\}$

(b) $W_2 = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2 \right\}$

(c) $W_3 = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0 \right\}$

(d) $W_4 = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0 \right\}$

(e) $W_5 = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1 \right\}$

(f) $W_6 = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \right\}$

Solution:

(a) W_1 is a subspace of $\mathbb{R}^3$:

- $(0,0,0) \in W_1$ as $0 = 3 \cdot 0$ and $0 = -0$
- For all $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_1$, we have $a_1 = 3a_2$, $a_3 = -a_2$, $b_1 = 3b_2$, $b_3 = -b_2$, and so $(a_1 + b_1) = 3(a_2 + b_2)$ and $a_3 + b_3 = -(a_2 + b_2)$, which implies that $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \in W_1$
- For all $(a_1, a_2, a_3) \in W_1$ and $c \in \mathbb{R}$, we have $a_1 = 3a_2$, $a_3 = -a_2$, and so $ca_1 = 3(ca_2)$ and $ca_3 = -(ca_2)$, which implies that $c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \in W_1$

(b) W_2 is not a subspace of $\mathbb{R}^3$: $(0,0,0) \notin W_2$ as $0 \neq 2 = 0 + 2$

(c) W_3 is a subspace of $\mathbb{R}^3$:

- $(0,0,0) \in W_3$ as $2 \cdot 0 - 7 \cdot 0 + 0 = 0$
- For all $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_3$, we have $2a_1 - 7a_2 + a_3 = 0$, $2b_1 - 7b_2 + b_3 = 0$, and so $2(a_1+b_1) - 7(a_2+b_2) + (a_3 + b_3) = (2a_1 - 7a_2 + a_3) + (2b_1 - 7b_2 + b_3) = 0$, which implies that $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \in W_3$
- For all $(a_1, a_2, a_3) \in W_3$ and $c \in \mathbb{R}$, we have $2a_1 - 7a_2 + a_3 = 0$, and so $2(ca_1) - 7(ca_2) + (ca_3) = c(2a_1 - 7a_2 + a_3) = 0$, which implies that $c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \in W_3$

(d) W_4 is a subspace of $\mathbb{R}^3$:

- $(0,0,0) \in W_4$ as $2 \cdot 0 - 7 \cdot 0 + 0 = 0$
- For all $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_4$, we have $a_1 - 4a_2 - a_3 = 0$, $b_1 - 4b_2 - b_3 = 0$, and so $(a_1+b_1) - 4(a_2+b_2) - (a_3+b_3) = (a_1 - 4a_2 - a_3) + (b_1 - 4b_2 - b_3) = 0$, which implies that $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \in W_4$
- For all $(a_1, a_2, a_3) \in W_4$ and $c \in \mathbb{R}$, we have $a_1 - 4a_2 - a_3 = 0$, and so $(ca_1) - 4(ca_2) - (ca_3) = c(a_1 - 4a_2 - a_3 = 0) = 0$, which implies that $c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \in W_4$

(e) W_5 is not a subspace of $\mathbb{R}^3$: $(0,0,0) \notin W_5$ as $0 + 2 \cdot 0 - 3 \cdot 0 = 0 \neq 1$

(f) W_6 is not a subspace of $\mathbb{R}^3$: $\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{3}}, 0\right), \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) \in W_6$ as $5\left(\frac{1}{\sqrt{5}}\right)^2 - 3\left(\frac{1}{\sqrt{3}}\right)^2 + 6(0)^2 = 5(0)^2 - 3\left(-\frac{1}{\sqrt{3}}\right)^2 + 6\left(\frac{1}{\sqrt{6}}\right)^2 = 0$, but $\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{3}}, 0\right) + \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) = \left(\frac{1}{\sqrt{5}}, 0, \frac{1}{\sqrt{6}}\right) \notin W_6$ as $5\left(\frac{1}{\sqrt{5}}\right)^2 - 3(0)^2 + 6\left(\frac{1}{\sqrt{6}}\right)^2 = 2 \neq 0$

Note

A rule of thumb to check whether a subset of $\mathbb{R}^n$ is subspace is to see if the defining constrains form a homogeneous linear system. Note that this does not (yet) constitute a proof. The rationale will be made clear in future lectures.

1.3.22. Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be fields. Prove that the set of all even functions in $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ and the set of all odd functions in $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ are subspaces of $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$.

Solution: Let $\vec{0} \in \mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ denote the zero function, i.e. $\vec{0}(t) = 0_{\mathbb{F}_2}$ for all $t \in \mathbb{F}_1$. Denote also the set of all even functions in $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ as $\mathcal{E}$, and the set of all odd functions in $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$ as $\mathcal{O}$

(a)
1. $\vec{0} \in \mathcal{E}$ as for all $t \in \mathbb{F}_1$, $\vec{0}(-t) = 0 = \vec{0}(t)$
2. Let $f, g \in \mathcal{E}$. Then $f(t) = f(-t)$, $g(t) = g(-t)$ for all $t \in \mathbb{F}_1$, so $(f+g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f+g)(t)$ for all $t \in \mathbb{F}_1$, hence $f + g \in \mathcal{E}$
3. Let $f \in \mathcal{E}$ and $a \in \mathbb{F}_2$. Then $(af)(-t) = af(-t) = af(t) = (af)(t)$ for all $t \in \mathbb{F}_1$, so $af \in \mathcal{E}$

Thus $\mathcal{E}$ is a subspace of $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$

(b)
1. $\vec{0} \in \mathcal{O}$ as for all $t \in \mathbb{F}_1$, $\vec{0}(-t) = 0 = -0 = -\vec{0}(t)$
2. Let $f, g \in \mathcal{O}$. Then $-f(t) = f(-t)$, $-g(t) = g(-t)$ for all $t \in \mathbb{F}_1$, so $(f+g)(-t) = f(-t) + g(-t) = -(f(t) + g(t)) = -(f+g)(t)$ for all $t \in \mathbb{F}_1$, hence $f + g \in \mathcal{O}$
3. Let $f \in \mathcal{O}$ and $a \in \mathbb{F}_2$. Then $(af)(-t) = af(-t) = -af(t) = -(af)(t)$ for all $t \in \mathbb{F}_1$, so $af \in \mathcal{O}$

Thus $\mathcal{O}$ is a subspace of $\mathcal{F}(\mathbb{F}_1, \mathbb{F}_2)$.

1.3.28. Let $\mathbb{F}$ be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from $\mathbb{F}$ is a subspace of $M_{n\times n}(\mathbb{F})$. Now assume that $\mathbb{F}$ is not of characteristic 2, and let W_2 be the subspace of $M_{n\times n}(\mathbb{F})$ consisting of all symmetric $n \times n$ matrices. Prove that $M_{n\times n}(\mathbb{F}) = W_1 \oplus W_2$.

Solution:

(a)
- Let $0_{n\times n} \in M_{n\times n}(\mathbb{F})$ denote the $n \times n$ zero matrix. Then $(0_{n\times n})_{ji} = 0 = -(0_{n\times n})_{ij}$ for all $i, j \in \{ 1, \ldots, n \}$. Hence $0_{n\times n} \in W_1$.
- Let $A, B \in W_1$. Then $A_{ji} = -A_{ij}$, $B_{ji} = -B_{ij}$ for all $i, j \in \{ 1, \ldots, n \}$. So for all $i, j \in \{ 1, \ldots, n \}$, $(A + B)_{ji} = A_{ji} + B_{ji} = -(A_{ij} + B_{ij}) = -(A + B)_{ij}$, hence $A + B \in W_1$
- Let $A \in W_1$, $c \in \mathbb{F}$. Then $A_{ji} = -A_{ij}$ for all $i, j \in \{ 1, \ldots, n \}$. So for all $i, j \in \{ 1, \ldots, n \}$, $(cA)_{ji} = cA_{ji} = -cA_{ij} = -(cA)_{ij}$, hence $cA \in W_1$

So W_1 is a subspace of $M_{n\times n}(\mathbb{F})$.

(b) Since $\mathbb{F}$ is not of characteristic 2, $2 = 1 + 1 \neq 0$ and $\frac{1}{2} \in \mathbb{F}$.

- Let $A \in M_{n\times n}(\mathbb{F})$. For $A_1 = \frac{1}{2}(A - A^\top), A_2 = \frac{1}{2}(A + A^\top) \in M_{n\times n}(\mathbb{F})$, we have $A_1^\top = \frac{1}{2}(A - A^\top)^\top = \frac{1}{2}(A^\top - A) = -A_1$, $A_2^\top = \frac{1}{2}(A + A^\top)^\top = \frac{1}{2}(A^\top + A) = A_2$. So $A_1 \in W_1$, $A_2 \in W_2$. By definition, we have $A_1 + A_2 = \frac{1}{2}(A - A^\top) + \frac{1}{2}(A + A^\top) = (1 + 1) \cdot \frac{1}{2}A = A$, so $A \in W_1 + W_2$.
  As A is arbitrary, $M_{n\times n}(\mathbb{F}) \subseteq W_1 + W_2$.
  Since $W_1, W_2 \subseteq M_{n\times n}(\mathbb{F})$, we have $W_1 + W_2 \subseteq M_{n\times n}(\mathbb{F})$. Thus $M_{n\times n}(\mathbb{F}) = W_1 + W_2$.
- Since W_1, W_2 are subspace of $M_{n\times n}(\mathbb{F})$, we have $\{0_{n\times n}\} \subseteq W_1 \cap W_2$.
  Let $A \in W_1 \cap W_2$. Then $A^\top = A$ and $A^\top = -A$. So $A = \frac{1}{2}(1 + 1) \cdot A^\top = \frac{1}{2}(A^\top - A^\top) = 0_{n\times n}$. As A is arbitrary, $W_1 \cap W_2 \subseteq \{0_{n\times n}\}$.
  Thus $W_1 \cap W_2 = \{0_{n\times n}\}$.

Therefore, $M_{n\times n}(\mathbb{F}) = W_1 \oplus W_2$.

Note

The characteristic of $\mathbb{F}$ not being 2 is necessary for the second part: in a field of characteristic 2, skew-symmetric matrices are exactly those that are symmetric as $-1 = 1$, but not every matrix is symmetric.

Please note also the similarity between this question and the previous one (Question 1.3.22).

1.3.30. Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

Solution:

(a) Suppose $V = W_1 \oplus W_2$. Then $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

Since $V = W_1 + W_2$, every vector $v \in V$ can be written as $v = x_1 + x_2$ for some $x_1 \in W_1, x_2 \in W_2$. It remains to show that this decomposition is unique.

Let $v \in V$ such that $v = x_1 + x_2 = y_1 + y_2$ with $x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$. Then $x_1 - y_1 = y_2 - x_2$. Since W_1, W_2 are subspaces of V, we have $x_1 - y_1 \in W_1$ and $y_2 - x_2 \in W_2$. So $x_1 - y_1 = y_2 - x_2 \in W_1 \cap W_2 = \{0\}$. This implies that $x_1 - y_1 = y_2 - x_2 = 0$, and so $x_1 = y_1$, $x_2 = y_2$.

Thus every vector in V can be uniquely written as $x_1 + x_2$ where $x_1 \in W_1, x_2 \in W_2$.

(b) Suppose each vector in V can be uniquely written as $x_1 + x_2$ where $x_1 \in W_1, x_2 \in W_2$.

Since every vector in $v \in V$ can be written as $v = x_1 + x_2$ for some $x_1 \in W_1, x_2 \in W_2$, we have $V \subseteq W_1 + W_2$. As W_1, W_2 are subspaces of V, we trivially also have $W_1 + W_2 \subseteq V$, so $V = W_1 + W_2$.

Trivially $\{0\} \subseteq W_1 \cap W_2$. Let $v \in W_1 \cap W_2$. Then $v = v + 0 = 0 + v$ as $0, v \in W_1 \cap W_2$. By the uniqueness of the decomposition, we must have $v = 0$ and $0 = v$. As v is arbitrary, this implies that $W_1 \cap W_2 \subseteq \{0\}$, so $W_1 \cap W_2 = \{0\}$.

Thus $V = W_1 \oplus W_2$.

Therefore, $V = W_1 \oplus W_2$ if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

Note

This proposition implies that direct sum is, in some sense, a natural way of combining subspaces. Different from join (span of unions, see Question 1.3.23 and Question 1.4.14, and the notes therein), direct sum concerns less about ordering (by inclusion) and more on the structure.

1.4.1. Label the following statements as true or false.

(a) The zero vector is a linear combination of any nonempty set of vectors.

(b) The span of $\emptyset$ is $\emptyset$.

(c) If S is a subset of a vector space V, then Span(S) equals the intersection of all subspaces of V that contain S.

(d) In solving a system of linear equations, it is permissible to multiply an equation by any constant.

(e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another.

(f) Every system of linear equations has a solution.

Solution:

(a) True

(b) False. Span($\emptyset$) $= \{0\}$ is the zero vector space

(c) True. Note that Span(S) is the smallest subspace of V that contains S, and intersections of subspaces is still a subspace. The proof is an easy exercise.

(d) False. Note that multiplying an equation by zero is prohibited (as it is not a reversible operation and may lead to lose of information, usually results in extraneous solutions)

(e) True

(f) False

1.4.4. For each list of polynomials in $\mathsf{P}_3(\mathbb{R})$, determine whether the first polynomial can be expressed as a linear combination of the other two.

(a) $x^3 - 3x + 5$, $x^3 + 2x^2 - x + 1$, $x^3 + 3x^2 - 1$

(b) $4x^3 + 2x^2 - 6$, $x^3 - 2x^2 + 4x + 1$, $3x^3 - 6x^2 + x + 4$

(c) $-2x^3 - 11x^2 + 3x + 2$, $x^3 - 2x^2 + 3x - 1$, $2x^3 + x^2 + 3x - 2$

(d) $x^3 + x^2 + 2x + 13$, $2x^3 - 3x^2 + 4x + 1$, $x^3 - x^2 + 2x + 3$

(e) $x^3 - 8x^2 + 4x$, $x^3 - 2x^2 + 3x - 1$, $x^3 - 2x + 3$

(f) $6x^3 - 3x^2 + x + 2$, $x^3 - x^2 + 2x + 3$, $2x^3 - 3x + 1$

Solution:

(a) $x^3 - 3x + 5$ is a linear combination of the other two polynomials: $x^3 - 3x + 5 = 3 \cdot (x^3 + 2x^2 - x + 1) - 2 \cdot (x^3 + 3x^2 - 1)$

(b) $4x^3 + 2x^2 - 6$ is not a linear combination of the other two polynomials. Suppose $4x^3 + 2x^2 - 6 = a \cdot (x^3 - 2x^2 + 4x + 1) + b \cdot (3x^3 - 6x^2 + x + 4) = (a + 3b)x^3 - 2(a + 3b)x^2 + (4a + b)x + (a + 4b)$ for some $a, b \in \mathbb{R}$. Then $a + 3b = 4$, $2(a + 3b) = 2$, $4a + b = 0$, $a + 4b = -6$. However, this system does not have any solution. So $4x^3 + 2x^2 - 6$ is not a linear combination of $x^3 - 2x^2 + 4x + 1$ and $3x^3 - 6x^2 + x + 4$

(c) $-2x^3 - 11x^2 + 3x + 2$ is a linear combination of the other two polynomials: $-2x^3 - 11x^2 + 3x + 2 = 4 \cdot (x^3 - 2x^2 + 3x - 1) - 3 \cdot (2x^3 + x^2 + 3x - 2)$

(d) $x^3 + x^2 + 2x + 13$ is a linear combination of the other two polynomials: $x^3 + x^2 + 2x + 13 = -2 \cdot (2x^3 - 3x^2 + 4x + 1) + 5 \cdot (x^3 - x^2 + 2x + 3)$

(e) $x^3 - 8x^2 + 4x$ is not a linear combination of the other two polynomials. Suppose $x^3 - 8x^2 + 4x = a \cdot (x^3 - 2x^2 + 3x - 1) + b \cdot (x^3 - 2x + 3) = (a + b)x^3 - 2ax^2 + (3a - 2b)x + (-a + 3b)$ for some $a, b \in \mathbb{R}$. Then $a + b = 1$, $-2a = -8$, $3a - 2b = 4$, $-a + 3b = 0$. However, this system does not have any solution. So $x^3 - 8x^2 + 4x$ is not a linear combination of $x^3 - 2x^2 + 3x - 1$ and $x^3 - 2x + 3$

(f) $6x^3 - 3x^2 + x + 2$ is not a linear combination of the other two polynomials. Suppose $6x^3 - 3x^2 + x + 2 = a \cdot (x^3 - x^2 + 2x + 3) + b \cdot (2x^3 - 3x + 1) = (a + 2b)x^3 - ax^2 + (2a - 3b)x + (3a + b)$ for some $a, b \in \mathbb{R}$. Then $a + 2b = 6$, $-a = -3$, $2a - 3b = 1$, $3a + b = 2$. However, this system does not have any solution. So $6x^3 - 3x^2 + x + 2$ is not a linear combination of $x^3 - x^2 + 2x + 3$ and $2x^3 - 3x + 1$

1.4.12. Show that a subset W of a vector space V is a subspace of V if and only if Span(W) $= W$.

Solution: Suppose W is a subspace of V. By the property of span, every subspace of V that contains W also contains $\text{Span}(\ W\ )$. Since $W \subseteq W$ and W is a subspace of V, we have $\text{Span}(\ W\ ) \subseteq W$. Trivially, we have $W \subseteq \text{Span}(\ W\ )$, so $W = \text{Span}(\ W\ )$.

Suppose $\text{Span}(\ W\ ) = W$. By the property of span, $W = \text{Span}(\ W\ )$ is a subspace of V.

Therefore, W is a subspace of V if and only if $\text{Span}(\ W\ ) = W$.

1.4.13. Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{Span}(\ S_1\ ) \subseteq \text{Span}(\ S_2\ )$. In particular, if $S_1 \subseteq S_2$ and $\text{Span}(\ S_1\ ) = V$, deduce that $\text{Span}(\ S_2\ ) = V$.

Solution:

(a) By the property of span, $S_2 \subseteq \text{Span}(\ S_2\ )$, so $S_1 \subseteq \text{Span}(\ S_2\ )$. As $\text{Span}(\ S_2\ )$ is a subspace of V, we have $\text{Span}(\ S_1\ ) \subseteq \text{Span}(\ S_2\ )$ as $\text{Span}(\ S_1\ )$ is the smallest subspace of V that contains S_1.

(b) Suppose $S_1 \subseteq S_2 \subseteq V$ and $\text{Span}(\ S_1\ ) = V$. Trivially, we have $\text{Span}(\ S_2\ ) \subseteq V$. By the previous part, we have $V = \text{Span}(\ S_1\ ) \subseteq \text{Span}(\ S_2\ )$, so $\text{Span}(\ S_2\ ) = V$.