THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A Linear Algebra II, 1st Term, 2022-23
Suggested Solution

1. (*20 points*) Label each statement as TRUE or FALSE. Moreover, give detailed reasons if your answer is FALSE.

   (a) Every change of coordinate matrix is invertible.

   (b) The sum of two eigenvectors of a linear operator T is always an eigenvector of T.

   (c) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

   (d) There exists a linear operator T on the vector space V that has no T-invariant subspace.

   (e) If T is a linear operator on a finite-dimensional vector space V and W is a T-invariant subspace of V, then the characteristic polynomial of T_W divides the characteristic polynomial of T.

Sol: (a) True.

   (b) False. Consider $T(x, y) = (x, 2y)$ and $T \in \mathcal{L}(\mathbb{R}^2)$. $(1, 0)$ and $(0, 1)$ are two eigenvectors but $(1, 1)$ is not.

   (c) False. Consider the identity transformation in $\mathbb{R}^2$. $(1, 0)$ and $(0, 1)$ are two linearly independent eigenvectors corresponding to the eigenvalue 1.

   (d) False. $\{0\}$ is always a T-invariant subspace.

   (e) True.

2. (*20 points*) Find the matrix presentation:

   (a) Define $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \gamma = \{1, x, x^2\}.$$

Compute $[T]_\beta^\gamma$.

(b) Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, ..., v_n\}$. Let $T : V \to V$ be a linear transformation such that

$$T(v_j) = v_j + v_{j-1}, \text{ for } j = 1, 2, ..., n,$$

where we set $v_0 = 0$. Compute $[T]_\beta$.

Sol:

(a) Direct calculation shows that

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1, \quad T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + x^2,$$

$$T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0, \quad T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 2x.$$

Then we conclude that

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(b) By definition, we have

$$T(v_1) = v_1 + v_0 = v_1 = 1 \cdot v_1,$$
$$T(v_k) = v_k + v_{k-1} = 1 \cdot v_{k-1} + 1 \cdot v_k \quad \text{for} \quad k = 2, \cdots, n$$

So we have $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

3. (*25 points*) Let $V = M_{2\times 2}(\mathbb{R})$, and define the linear operator T on V by

$$T(A) = A^t,$$

where A^t is the transpose of A. Test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

Sol: Let $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be an ordered basis of V. Then

$$[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of T is given by

$$\det([T]_\gamma - xI_4) = (x^2 - 1)(x - 1)^2 = (x - 1)^3(x + 1).$$

It splits over $\mathbb{R}$ and the eigenvalues of T are $1, -1$, with multiplicity $3, 1$ respectively. We check that

$$[T]_\gamma - I_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence $4 - \text{rank}(T - I_V) = 4 - 1 = 3$ which is the multiplicity of 1. We check that

$$[T]_\gamma + I_4 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

then $\dim(E_{-1}) = 1$ which is the multiplicity of -1. Therefore T is diagonalizable.

By computation, the null space of $[T]_\gamma - I_4$ is span by the linearly independent set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Therefore

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for the eigenspace E_1.

By direct calculation, the null space of $[T]_\gamma + I_4$ is span by the linearly independent set $\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Therefore $\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ is a basis for the eigenspace E_{-1}.

Combining the bases, we have

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

being an ordered basis for V consisting of eigenvectors of T. Hence $[T]_\beta$ is a diagonal matrix.

4. (*20 points*) Answer the following questions:

   (a) State without any proof the Cayley-Hamilton theorem you learned from the course.

   (b) Let A be an $n \times n$ matrix. Use the Cayley-Hamilton theorem to prove that

   $$\dim(\text{span}(\{I_n, A, A^2, \cdots\})) \leq n,$$

   where I_n is the $n \times n$ identity matrix. *Hint: Cayley-Hamilton theorem tells that A^n is a linear combination of $I_n, A, ..., A^{n-1}$.*

Sol:

(a) Let $T \in \mathcal{L}(V)$ with $\dim(V) < \infty$, and $f(t)$ be the c.p. of T. Then, T satisfies the characteristic equation in the sense that $f(T) = T_0$, i.e., $f(T)$ is a zero transformation.

(b) Let $U = \text{span}(\{I, \ldots, A_{n-1}\})$. Then $\dim U \leq n$.

To show the proposition, we show that $\text{span}(\{I, A, \ldots\}) = U$. By definition, $\text{span}(\{I, A, \ldots\}) \supseteq U$. It then suffices to show that $A^k \in U$ for all $k \in \mathbb{N}$. The case where $k < n$ is trivial from the definition of U.

Suppose there exists $l \geq n-1$ such that $I, A, \ldots, A^l \in U$. Let the characteristic polynomial of A be $p(t)$. Then $\deg p = n$. We may assume that $p(t) = \sum_{i=0}^{n} c_i t^i$ for some scalar $c_0, \ldots, c_n$ with $c_n = (-1)^n$. By Cayley-Hamilton theorem, $p(A) = \sum_{i=0}^{n} c_i A^i = c_0 I + \ldots + c_n A^n = 0$. So $A^n = \sum_{i=0}^{n-1} -\frac{c_i}{c_n} A^i$, $A^{l+1} = A^{l-n+1}A^n = \sum_{i=0}^{n-1} -\frac{c_i}{c_n} A^{l-n+1+i} \in U$ as $A^{l-n+1}, \ldots, A^l \in U$.

By induction, $A^k \in U$ for all $k \in \mathbb{N}$.

So $\text{span}(\{I, A, \ldots\}) = U$ and $\dim \text{span}(\{I, A, \ldots\}) = \dim U \leq n$.

5. (*15 points*) Let T be a linear operator on the vector space V with $\text{rank}(T) = k$. Prove that T has at most $k+1$ distinct eigenvalues. *Hint: Think about the otherwise case when T has at least $k+2$ distinct eigenvalues in which there should exist at least $k+1$ distinct nonzero eigenvalues.*

Sol: We prove by contradiction. Consider if T has at least $k+2$ distinct eigenvalues. Then at least $k+1$ of them are both distinct and nonzero. We denote them by $\lambda_1, \cdots, \lambda_{k+1}$. Also their corresponding eigenvectors are denoted by $v_1, \cdots, v_{k+1}$. Since the eigenvalues are distinct, $\{v_1, \cdots, v_{k+1}\}$ is linearly independent. Next we prove that $\text{span}(\{v_1, \cdots, v_{k+1}\}) = \text{span}(\{\lambda_1 v_1, \cdots, \lambda_{k+1} v_{k+1}\})$. It is direct to see that $\text{span}(\{v_1, \cdots, v_{k+1}\}) \supset \text{span}(\{\lambda_1 v_1, \cdots, \lambda_{k+1} v_{k+1}\})$. For any $x \in \text{span}(\{v_1, \cdots, v_{k+1}\})$, there exist $c_1, \cdots, c_{k+1}$ such that

$$\begin{aligned} x &= c_1 v_1 + \cdots + c_{k+1} v_{k+1} \\ &= \frac{c_1}{\lambda_1} \lambda_1 v_1 + \cdots + \frac{c_{k+1}}{\lambda_{k+1}} \lambda_{k+1} v_{k+1} \in \text{span}(\{\lambda_1 v_1, \cdots, \lambda_{k+1} v_{k+1}\}, \end{aligned}$$

since $\lambda_1, \cdots, \lambda_{k+1}$ are nonzero. Hence we have $\text{span}(\{v_1, \cdots, v_{k+1}\}) \subset \text{span}(\{\lambda_1 v_1, \cdots, \lambda_{k+1} v_{k+1}\})$, which yields that $\text{span}(\{v_1, \cdots, v_{k+1}\}) = \text{span}(\{\lambda_1 v_1, \cdots, \lambda_{k+1} v_{k+1}\})$. We have

$$\begin{aligned} k &= \dim(R(T)) \geq \dim(\text{span}(\{Tv_1, \cdots, Tv_{k+1}\})) \\ &= \dim(\text{span}(\{\lambda_1 v_1, \cdots, \lambda_{k+1} v_{k+1}\})) = k+1, \end{aligned}$$

which leads to contradiction.

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