Topic#15
Orthogonal complement

Def. V: i.p.s. with $\langle\cdot,\cdot\rangle$. $\emptyset \neq S \subset V$.

$$S^{\perp} \overset{def}{=} \{x \in V : \langle x, y\rangle = 0, \forall y \in S\}$$

is called the **orthogonal complement** of S.

Note: $S^{\perp}$ is a subspace of V. (prove this!) S need not be a subspace

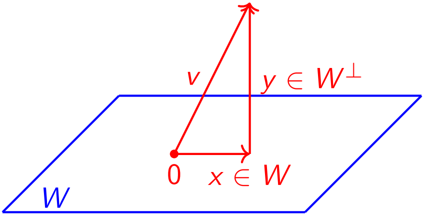
e.g.
1°. $\{0\}^{\perp} = V$, $V^{\perp} = \{0\}$.
2°. $V = \mathbb{R}^3$, $S = \{e_3\}$, then $S^{\perp}$ =span($\{e_1, e_2\}$) is the xy-plane.

Theorem. V: i.p.s with $\langle \cdot, \cdot \rangle$ (can be infinite-dim).
W: subspace of V, $\dim(W) < \infty$. Then,

$$\forall v \in V, \exists! x \in W \ \& \ y \in W^{\perp} \text{ s.t. } v = x + y.$$

Furthermore, if W has an orthonormal basis $\{w_1, \cdots, w_k\}$ then

$$x = \sum_{i=1}^{k} \langle v, w_i \rangle w_i.$$



Def. x is called the **orthogonal projection** of $v \in V$ on W.

Pf. • Existence: Let $v \in V$. Define

$$x \overset{def}{=} \sum_{i=1}^{k} \langle v, w_i \rangle w_i \in W, \quad y \overset{def}{=} v - x \in V.$$

Claim. $y \in W^{\perp}$.
Indeed, recall $W^{\perp} = \{u \in V : \langle u, w \rangle = 0, \forall w \in W\}$,
then it suffices to show $\langle y, w \rangle = 0, \forall w \in W$.
Let $w \in W =$span$(\{w_1, \cdots, w_k\})$, then $w = \sum_{j=1}^{k} a_j w_j$, and

$$\begin{aligned}
\langle y, w \rangle &= \langle v - \sum_{i=1}^{k} \langle v, w_i \rangle w_i, \sum_{j=1}^{k} a_j w_j \rangle \\
&= \sum_{j=1}^{k} \bar{a}_j \langle v, w_j \rangle - \sum_{i=1}^{k} \sum_{j=1}^{k} \bar{a}_j \langle v, w_i \rangle \langle w_i, w_j \rangle \\
&= \sum_{j=1}^{k} \bar{a}_j \langle v, w_j \rangle - \sum_{j=1}^{k} \bar{a}_j \langle v, w_j \rangle = 0. \quad \square
\end{aligned}$$

- Uniqueness: Let $v = x + y = x' + y'$ for $x, x' \in W$, $y, y' \in W^{\perp}$.

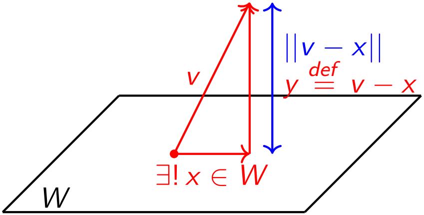
$\therefore x - x' = y - y' \in W \cap W^{\perp} = \{0\}$(why?). $\therefore x = x', y = y'$. $\square$

Hint of 'why': let $y \in W \cap W^{\perp}$, to show $y = 0$.

See $y \in W$ and $y \perp$ any vector in W.

So we get $\langle y, y \rangle = 0 \Rightarrow y = 0_v$. $\square$

Remark:



> **Claim.** Let $v \in V$, and $x \overset{def}{=} \sum_{i=1}^{k} \langle v, w_i \rangle w_i \in W$. Then
>
> $$\|v-w\| \geq \|v-x\|, \; \forall\, w \in W,$$
>
> where "=" holds iff $w = x$. Namely, for all vectors in W, x is the unique vector that is "closest" to v.

Pf. $\|v-w\|^2 = \|(x+y)-w\|^2 = \|(x-w)+y\|^2$ $y \overset{def}{=} v-x$

$$\overset{Pythagorian}{=} \|x-w\|^2 + \|y\|^2 \; (\because\; x-w \in W, y \in W^{\perp}) \geq \|y\|^2.$$

"=" holds $\Leftrightarrow \|x-w\| = 0 \Leftrightarrow x = w$. □

Application: Let $v = x + y$ for $x \in W, y \in W^{\perp}$, then

$$\|v\|^2 = \langle v, v\rangle = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2. \ \therefore \|v\|^2 \geq \|x\|^2.$$

Note: $x = \sum_{i=1}^{k} \langle v, w_i\rangle w_i$, then $\|x\|^2 = \sum_{i=1}^{k} |\langle v, w_i\rangle|^2$(prove it!).

$$\therefore \|v\|^2 \geq \sum_{i=1}^{k} |\langle v, w_i\rangle|^2 \ \ \forall v \in V$$ (Beseel's inequality)

For instance, $V = H([0, 2\pi])$.
Recall that $S = \{e^{int} : n = 0, \pm 1, \cdots\}$ is an orthonormal set of H.
Consider $W \overset{def}{=} \text{span}(S)$ =span($\{e^{int} : n = 0, \pm 1, \cdots, \pm k\}$).
Note $\dim(W) = 2k + 1$. For $\forall f \in H$, then

$$\sum_{n=-k}^{k} |\langle f, e^{int}\rangle|^2 \leq \|f\|^2$$

In particular, for $f(t) = t$,

RHS:
$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} t^2 dt = \frac{4}{3}\pi^2$.

LHS:
$n = 0$: $\langle f, e^{i0t}\rangle = \langle f, 1\rangle = \frac{1}{2\pi}\int_0^{2\pi} tdt = \pi$.
$n = \pm 1, \pm 2, \cdots$: $\langle f, e^{int}\rangle = \frac{1}{2\pi}\int_0^{2\pi} t\overline{e^{int}}dt = \cdots = -\frac{1}{in}$.

$\therefore$ Beseel's inequality gives

$$\frac{4}{3}\pi^2 \geq |\pi|^2 + \sum_{n=\pm 1,\cdots,\pm k} |-\frac{1}{in}|^2 = \pi^2 + 2\sum_{i=1}^{k}\frac{1}{n^2}$$
$$\therefore \sum_{n=1}^{k}\frac{1}{n^2} \leq \frac{1}{6}\pi^2, k = 1, 2, \cdots$$

Let $k \to \infty$,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \leq \frac{1}{6}\pi^2.$$

Exercise. Try any other $f \in H$.

Remark:

V: i.p.s. with $\langle \cdot, \cdot \rangle$.
$\beta \subset V$: othonormal (possibly infinite)
Let $v \in V$. We may compute

$$\langle v, x \rangle, x \in \beta$$

which is called the **Fourier coefficients** of v relative to β.
(Above, take the $\beta = S$.)

Last goal of this topic:

Can a finite orthonormal (should be l. indep.) set be extended to an orthonormal basis ?

Theorem. let V be an i.p.s with $\dim(V) = n$. Then

(1) An othonormal set $S = \{v_1, \cdots, v_k\}$ can be extended to an orthonormal basis $\{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ for V. Furthermore, let $W =$span(S), then

$$S_1 = \{v_{k+1}, \cdots, v_n\} \text{ is an orthonormal basis for } W^{\perp}.$$

(2) For any subspece W of V,

$$V = W \bigoplus W^{\perp} and \dim(V) = \dim(W) + \dim(W^{\perp}).$$

Pf. (1) $S=\{v_1,\cdots,v_k\}$: orthonormal
$\xrightarrow{extension}$ $\{\underset{\sim\sim\sim\sim}{v_1,\cdots,v_k}, w_{k+1},\cdots,w_n\}$: o.b. for V

$\xrightarrow{G.-S.}$ $\{\underset{\sim\sim\sim\sim}{v_1,\cdots,v_k}, w'_{k+1},\cdots,w'_n\}$: orthogonal o.b. for V

(The first k vectors must be still $v_1,\cdots,v_k$; see Ex. 8 of Sec. 6.2)
$\xrightarrow{normalizing}$ $\{v_1,\cdots,v_k,v_{k+1},\cdots,v_n\}$: orthonormal o.b. for V.

Let W =span(S)=span($\{v_1,\cdots,v_k\}$), to show:

$$S_1 \stackrel{def}{=} \{v_{k+1},\cdots,v_n\} \text{ is an orthonormal basis for } W^\perp.$$

Indeed,
1°. $S_1 \subset W^\perp$ & S_1 is orthonormal.
2°. $W^\perp \subset \text{span}(S_1)$.
In fact, let $x \in W^\perp \subset V$. Note: $x=\sum_{i=1}^n \langle x, v_i\rangle v_i$. Then,

$$x \in W^\perp \Rightarrow \langle x, v_i\rangle = 0, (i=1,\cdots,k; v_i \in W \Rightarrow x \perp v_i).$$

$\therefore x=\sum_{i=k+1}^n \langle x, v_i\rangle v_i \in$span$(S_1)$. $\therefore W^\perp \subset$span$(S_1)$. □

(2) Let W be a subspace of V, then W is a finite-dim i.p.s., and hence it has an orthonormal basis $\{v_1, v_2, \cdots, v_k\}$. By (1),

$$\begin{aligned}\dim(V) &= n \\ &= k + (n-k) \\ &= \dim(W) + \dim(W^{\perp}).\end{aligned}$$

$\square\square$