MATH 2028 Honours Advanced Calculus II
2022-23 Term 1
Problem Set 4

due on Oct 14, 2022 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: Throughout this problem set, we use (r,θ), (r,θ,z) and (ρ,ϕ,θ) to denote the polar, cylindrical and spherical coordinates respectively.

Problems to hand in

1. Find the area enclosed by the cardioid in $\mathbb{R}^2$ expressed in polar coordinates as $r = 1+\cos\theta$.
2. Evaluate the iterated integral
$$\int_0^1 \int_y^1 \frac{xe^x}{x^2+y^2}\,dxdy.$$
3. Find the volume of the region lying above the plane $z=a$ and inside the sphere $x^2+y^2+z^2=4a^2$ by integrating in cylindrical coordinates and spherical coordinates.
4. Find the volume of the region in $\mathbb{R}^3$ bounded by the cylinders $x^2+y^2=1$, $y^2+z^2=1$, and $x^2+z^2=1$.

Suggested Exercises

1. Let $\Omega \subset \mathbb{R}^2$ be the region bounded below by $y=1$ and above by $x^2+y^2=4$. Evaluate
$$\int_\Omega (x^2+y^2)^{-3/2}\,dA.$$
2. Let $\Omega \subset \mathbb{R}^2$ be the annular region bounded by $x^2+y^2=1$ and above by $x^2+y^2=2$. Evaluate $\int_\Omega y^2\,dA$.
3. Find the volume of the region in $\mathbb{R}^3$ bounded above by $z=2$ and below by $z=x^2+y^2$.
4. Find the volume of the region $\mathbb{R}^3$ inside both $x^2+y^2=1$ and $x^2+y^2+z^2=2$.
5. Find the volume of a right circular cone of base radius a and height h by integrating in cylindrical coordinates and spherical coordinates.
6. Let $\Omega \subset \mathbb{R}^3$ be the region bounded below by the sphere $x^2+y^2+z^2=2z$ and above by the sphere $x^2+y^2+z^2=1$. Evaluate the integral
$$\int_\Omega \frac{z}{(x^2+y^2+z^2)^{3/2}}\,dV.$$

7. (a) Let $\epsilon > 0$ be fixed. Show that there is a C^∞ function $g : \mathbb{R} \to [0, 1]$ such that $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq \epsilon$.

   (b) Let $\Omega \subset \mathbb{R}^n$ be an open set and $K \subset \Omega$ be a compact subset. Prove that there exists a C^∞ function $f : \Omega \to [0, 1]$ such that $f(x) = 1$ for all $x \in K$.

Challenging Exercises

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded subset with measure zero $\partial\Omega$. Show that for any $\epsilon > 0$, there exists a compact subset $K \subset \Omega$ such that ∂K has measure zero and $\mathrm{Vol}(\Omega \setminus K) < \epsilon$.

2. (a) Let $S \subset \mathbb{R}^n$ be an arbitrary subset and $x_0 \in S$. We say that a function $f : S \to \mathbb{R}$ is differentiable at x_0 of class C^1 if there exists a C^1 function $g : U \to \mathbb{R}$ defined in a neighborhood U of x_0 in $\mathbb{R}^n$ such that $g = f$ on $U \cap S$. Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function whose support lies in U. Show that the function $h : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \varphi(x)g(x) & \text{when } x \in U \\ 0 & \text{when } x \notin \mathrm{spt}(\varphi) \end{cases}$$

   is a well-defined C^1 function on $\mathbb{R}^n$.

   (b) Prove the following statement: if $f : S \to \mathbb{R}$ is differentiable of class C^1 at each $x_0 \in S$, then f may be extended to a C^1 function $h : \Omega \to \mathbb{R}$ defined on an open subset Ω containing S.