

MATH 2028 Honours Advanced Calculus II
2022-23 Term 1
Problem Set 3

due on Oct 5, 2022 (Wednesday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: Throughout this problem set, we use R to denote a rectangle in \mathbb{R}^n . When we write $R = A \times B$, then we mean $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^k$ are rectangles with $n = m + k$.

Problems to hand in

1. Evaluate the following integrals:

(a) $\int_R \frac{y}{x} dV$ where $R = [1, 3] \times [2, 4]$

(b) $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y dx dy$

(c) $\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{1+x^3} dx dy$

2. Find the volume of the region in \mathbb{R}^3 bounded below by the xy -plane, above by $z = y$, and on the sides by $y = 4 - x^2$.

3. Let $f : R = [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a continuous function such that $\frac{\partial f}{\partial y}$ is continuous on R . Define $G : [c, d] \rightarrow \mathbb{R}$ such that

$$G(y) := \int_a^b f(x, y) dx.$$

(a) Show that G is continuous on $[c, d]$.

(b) Prove that G is differentiable on (c, d) and $G'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y) dx$.

4. Let $f : R = [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q}, \\ 2x & \text{if } y \notin \mathbb{Q}. \end{cases}$$

(a) Prove that f is NOT integrable on R .

(b) Show that each iterated integral $\int_0^1 \int_0^1 f(x, y) dx dy$ and $\int_0^1 \int_0^1 f(x, y) dy dx$ exist and compute their values.

Suggested Exercises

1. Evaluate the following integrals:

(a) $\int_R \frac{x}{x^2+y} dV$ where $R = [0, 1] \times [1, 3]$

- (b) $\int_0^1 \int_{x^2}^x \frac{x}{1+y^2} dy dx$
 (c) $\int_0^1 \int_{\sqrt{y}}^1 e^{y/x} dx dy$

2. Find the volume of the region in \mathbb{R}^3 bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.
 3. Let $\Omega \subset \mathbb{R}^3$ be the portion of the cube $[0, 1] \times [0, 1] \times [0, 1]$ lying above the plane $y + z = 1$ and below the plane $x + y + z = 2$. Evaluate the integral $\int_{\Omega} x dV$.
 4. Let $f : \Omega \rightarrow \mathbb{R}$ be a C^2 function¹ on an open subset $\Omega \subset \mathbb{R}^2$. Use Fubini's Theorem to prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ everywhere in Ω .
 5. Let $f : R = [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a continuous function. Define another function $F : R \rightarrow \mathbb{R}$ such that

$$F(x, y) := \int_{[a,x] \times [c,y]} f dV.$$

Compute $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in the interior of R .

6. Let $f(x) = \left(\int_0^x e^{-t^2} dt \right)^2$ and $g(x) = \int_0^1 (t^2 + 1)^{-1} e^{-x^2(t^2+1)} dt$.
 (a) Prove that $f(x) + g(x) = \frac{\pi}{4}$ for all $x \geq 0$.
 (b) Use (a) to show that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.
 7. Let P and Q be bounded subsets of \mathbb{R}^3 whose boundaries have measure zero. For each $z_0 \in \mathbb{R}$, we define the following subsets of \mathbb{R}^2 :

$$P_{z_0} := \{(x, y) \in \mathbb{R}^2 \mid (x, y, z_0) \in P\},$$

$$Q_{z_0} := \{(x, y) \in \mathbb{R}^2 \mid (x, y, z_0) \in Q\}.$$

Suppose for EACH $z_0 \in \mathbb{R}$, the boundaries ∂P_{z_0} and ∂Q_{z_0} have measure zero in \mathbb{R}^2 and we have $\text{Area}(P_{z_0}) = \text{Area}(Q_{z_0})$. Prove that $\text{Vol}(P) = \text{Vol}(Q)$.

8. Let $f : R = A \times B \rightarrow \mathbb{R}$ be a bounded integrable function. Suppose $g : A \rightarrow \mathbb{R}$ is a function satisfying

$$\int_{\underline{B}} f(x, y) dy \leq g(x) \leq \int_{\overline{B}} f(x, y) dy$$

for all $x \in A$. Prove that g is integrable over A and $\int_A g(x) dx = \int_R f dV$.

Challenging Exercises

1. (a) Let $C \subset R = A \times B$ be a set of content zero in \mathbb{R}^n . Let A' be the set of all $x \in A$ such that the set $\{y \in B \mid (x, y) \in C\}$ does NOT have content zero in \mathbb{R}^k . Prove that A' is a set of measure zero in \mathbb{R}^m .
 (b) Let $C \subset R = [0, 1] \times [0, 1]$ be the set consisting of all $(x, y) \in R$ such that $x = \frac{p}{q} \in \mathbb{Q}$, where $p, q \in \mathbb{N}$ are coprime, and $y \in [0, \frac{1}{q}]$. Prove that C has content zero in \mathbb{R}^2 but the subset A' as defined in (a) does NOT have content zero in \mathbb{R} .

¹Recall that a function f is C^k if all the partial derivatives up to order k exist and are continuous.