MATH 2028 Honours Advanced Calculus II
2022-23 Term 1
Problem Set 2

due on Sep 28, 2022 (Wednesday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through CUHK Blackboard on/before the due date. Please remember to write down your name and student ID. You can refer to the webpage under "Useful Links" below about how to submit assignments through Blackboard. **No late homework will be accepted.**

Notations: Throughout this problem set, we use R to denote a rectangle in $\mathbb{R}^n$, and $B_\delta(p) \subset \mathbb{R}^n$ to denote the open ball of radius δ centered at p.

Problems to hand in

1. (a) Prove that any content zero subset $A \subset \mathbb{R}^n$ must also have measure zero.
   (b) Give an example of a measure zero subset $A \subset \mathbb{R}^2$ which does not have content zero.
   (c) Prove that if $A \subset \mathbb{R}^n$ is compact [1] and has measure zero, then A has content zero.
   (d) Suppose $\{A_i\}_{i=1}^\infty$ is a sequence of measure zero subsets in $\mathbb{R}^n$. Show that $\cup_{i=1}^\infty A_i$ also has measure zero.

2. Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ be the function

$$f(x,y) = \begin{cases} 1/q & \text{if } x, y \in \mathbb{Q} \text{ and } y = p/q \text{ where } p, q \in \mathbb{N} \text{ are coprime,} \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that f is integrable and $\int_R f \, dV = 0$.

3. Let $f : \Omega \to \mathbb{R}$ be a bounded continuous function defined on a bounded subset $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ has measure zero. Suppose Ω is path-connected, i.e. for any $p, q \in \Omega$, there exists a continuous path $\gamma(t) : [0,1] \to \Omega$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Prove that there exists some $x_0 \in \Omega$ such that

$$\int_\Omega f \, dV = f(x_0)\mathrm{Vol}(\Omega).$$

Suggested Exercises

1. Let $f : R \to \mathbb{R}$ be a bounded integrable function. Suppose p is an interior point of R at which f is continuous. Prove that

$$\lim_{\delta \to 0^+} \frac{1}{\mathrm{Vol}(B_\delta(p))} \int_{B_\delta(p)} f \, dV = f(p).$$

2. Let $f : R \to \mathbb{R}$ be a bounded integrable function. Prove that $|f|$ is also integrable on R and $\left| \int_R f \, dV \right| \leq \int_R |f| \, dV$.

[1] A subset A is compact if any open cover of A has a finite subcover. The Heine-Borel Theorem says that a subset in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

3. (a) Let $A \subset \mathbb{R}^n$ be a content zero subset. Prove that A must be bounded. Moreover, show that ∂A has measure zero and $\text{Vol}(A) = 0$.

   (b) Let $B \subset \mathbb{R}^n$ be a bounded subset of measure zero. Suppose ∂B has measure zero. Prove that $\text{Vol}(B) = 0$.

4. (a) Show that the subset $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ has measure zero.

   (b) Show that $\mathbb{Q}^c \cap [0,1]$ does not have measure zero in $\mathbb{R}$.

5. Let $f : R \to \mathbb{R}$ be a bounded function. Suppose $f = 0$ except on a *closed* set B of measure zero. Prove that f is integrable and $\int_R f\, dV = 0$.

Challenging Exercises

1. The following exercise establishes the theorem that a bounded function $f : R \to \mathbb{R}$ is integrable if and only if f is continuous on R except on a set of measure zero. Let $f : R \to \mathbb{R}$ be a bounded function. For each $p \in R$ and $\delta > 0$, we define the *oscillation of* f *at* p as

$$o(f,p) = \lim_{\delta \to 0^+} \left(\sup_{x \in B_\delta(p) \cap R} f(x) - \inf_{x \in B_\delta(p) \cap R} f(x) \right).$$

   (a) Show that $o(f,p)$ is well-defined and non-negative. Prove that f is continuous at p if and only if $o(f,p) = 0$.

   (b) For any $\epsilon > 0$, let $D_\epsilon := \{p \in R \,:\, o(f,p) \geq \epsilon\}$. Show that D_ϵ is a closed subset and the set of discontinuities D of f is given as $D = \cup_{n=1}^\infty D_{1/n}$.

   (c) Suppose f is integrable on R. Prove that $D_{1/n}$ has content zero for any $n \in \mathbb{N}$. Hence, show that D has measure zero.

   (d) Suppose D has measure zero, prove that f is integrable on R.

2. This exercise requires some familiarity with linear algebra at the level of MATH 2040/2048.

   (a) Let $A \subset \mathbb{R}^n$ be a subset of content zero. Show that for any $\epsilon > 0$, there exists finitely many cubes [2] $C_1, \cdots, C_n$ such that $A \subset \cup_{i=1}^n C_i$ and $\sum_{i=1}^n \text{Vol}(C_i) < \epsilon$.

   (b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that $T(A)$ has content zero if $A \subset \mathbb{R}^n$ has content zero.

   (c) Let $F : U \to \mathbb{R}^n$ be a C^1 map from an open subset $U \subset \mathbb{R}^m$ where $m < n$. Prove that $F(A)$ has content zero (in $\mathbb{R}^n$) if $A \subset U$ is a compact subset.

[2] A cube is a rectangle with sides of equal length.