March 15, 2023
Week 9

2020B Adv. Cal II

A *parametric curve* is a continuous map: $[a,b] \to \mathbb{R}^n$, ($n=2,3$ is our main concern). When $n=3$,

$$\vec{c}(t) = (x(t), y(t), z(t)) \overset{\text{or}}{=} x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$
$$\vec{c}\,'(t) = (x'(t), y'(t), z'(t)) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$$

When $\vec{c}(t)$ describes the position of a moving particle at time t, $\vec{c}\,'(t)$ is its velocity at $\vec{c}(t)$ at t, and $|\vec{c}\,'(t)|$ is its speed.

$$|\vec{c}\,'(t)| = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}.$$

A parametric curve is *regular* if

(a) $\vec{c}\,'(t)$ exists and is continuous, and

(b) $\vec{c}\,'(t)$ is always a non-zero vector, ie, $|\vec{c}\,'(t)| > 0$, for all t.

For a regular parametric curve,

$$\hat{t} = \frac{\vec{c}\,'(t)}{|\vec{c}\,'(t)|}$$

is well-defined, call it the *tangent* (unit tangent vector) at $\vec{c}(t)$.

For curves in $\mathbb{R}^2$, we use

$$\vec{c}(t) = (x(t), y(t)) = x(t)\hat{i} + y(t)\hat{j}$$
$$\vec{c}\,'(t) = (x'(t)\ y'(t))$$
$$|\vec{c}\,'(t)| = \sqrt{x'^2(t) + y'^2(t)}.$$

e.g. the helix $\vec{c}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$, $t \in [0, 2\pi]$.

$$|\vec{c}\,'(t)| = |-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}|$$
$$= \sqrt{2} \quad \text{(constant speed)}$$

$\therefore$ it's a regular parametric curve.

e.g. the circle $\vec{c}(t) = (a\cos t, a\sin t)$, $t \in [0, 2\pi]$

$$= a\cos t\,\hat{i} + a\sin t\,\hat{j}$$

$$|\vec{c}'(t)| = |-a\sin t\,\hat{i} + a\cos t\,\hat{j}|$$

$$= a > 0$$

It's a regular parametric curve with constant speed. As t runs from 0 to 2π, the particle moves around the circle once in anticlockwise way.

A <u>curve</u> (or smooth curve more precise) is the image of a regular parametrization, i.e., a set C in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a curve if there is a regular parametrization $\vec{c}$ s.t.

$$C = \{\vec{c}(t) : t \in [a, b]\}.$$

e.g. Find a parametrization for the line segment from $\vec{u}$ to $\vec{v}$.

$$\vec{c}(t) = \vec{u} + t(\vec{v} - \vec{u}), \quad t \in [0, 1]$$

$$\vec{c}'(t) = \vec{v} - \vec{u},$$

$$|\vec{c}'(t)| = |\vec{v} - \vec{u}| > 0 \text{ regular.}$$

e.g. Let $f(x)$, $x \in [a, b]$, be a smooth function. Then its graph $\{(x, f(x)) : x \in [a, b]\}$ forms a smooth curve, a regular parametrization is

$$\vec{c}(x) = x\,\hat{i} + f(x)\,\hat{j}, \quad x \in [a, b]$$

$$\vec{c}'(x) = \hat{i} + f'(x)\,\hat{j}$$

$$|\vec{c}'(x)| = \sqrt{1 + f'^2(x)} > 0.$$

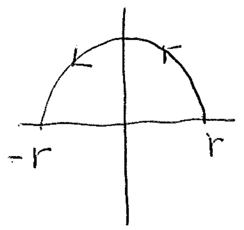
Let C be a curve and f a function on C. The <u>line integral</u> of f <u>along</u> C is defined to be

$$\underbrace{\int_C f\,ds}_{\text{notation}} = \int_a^b f(\vec{c}(t))\,|\vec{c}\,'(t)|\,dt,$$ where $\vec{c}(t)$ is a regular

parametrization of C. It can be shown that this integral is indept of the choice of parametrization.

- when $f \geq 0$, $\int_C f\,ds$ is the mass of the wire C with density f.
- when $f \equiv 1$, $\int_C ds$ is the <u>length</u> of C.

e.g. Find the length of the half-circle $x^2+y^2=r^2$, $y \geq 0$.



$\vec{c}(t) = r\cos t\,\hat{i} + r\sin t\,\hat{j}$, $t \in [0,\pi]$.

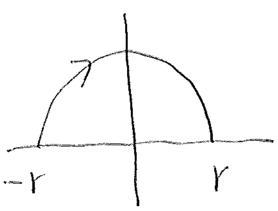
$\vec{c}(t)$ parametrises the half circle.

$\vec{c}\,'(t) = -r\sin t\,\hat{i} + r\cos t\,\hat{j}$

$|\vec{c}\,'(t)| = \left((-r\sin t)^2 + (r\cos t)^2\right)^{1/2} = r$

$\therefore$ length $= \int_C 1\,ds = \int_0^\pi r\,dt = \pi r$ #

Alternatively, $\vec{c}_1(x) = x\,\hat{i} + \sqrt{r^2-x^2}\,\hat{j}$, $x \in [-r,r]$



also parametrises the half circle.

$\vec{c}_1\,'(x) = \hat{i} + \dfrac{-x}{\sqrt{r^2-x^2}}\,\hat{j}$

$|\vec{c}_1\,'(x)| = \sqrt{1 + \left(\dfrac{-x}{\sqrt{r^2-x^2}}\right)^2} = \dfrac{r}{\sqrt{r^2-x^2}}$.

$$\text{length} = \int_{-r}^{r} \frac{r\,dx}{\sqrt{r^2-x^2}} = 2\int_0^r \frac{r\,dx}{\sqrt{r^2-x^2}} = 2r\int_0^{\pi/2} \frac{r\cos\theta\,d\theta}{\sqrt{r^2-r^2\sin^2\theta}},$$

$x = r\sin\theta$

$= \pi r$, the same as before. #

Step 1 Find a parametrization for C, $\vec{c}(t)$, $t\in[a,b]$.

Step 2 Calculate $\vec{c}\,'(t)$ and $|\vec{c}\,'(t)|$.

Step 3 Plug in, and integrate

$$\int_a^b f(\vec{c}(t))\,|\vec{c}\,'(t)|\,dt$$

A vector field is

$$\vec{F}(x,y,z) = (M(x,y,z),\ N(x,y,z),\ P(x,y,z))$$
$$= M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}, \quad n=3.$$

$$\left(\vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j},\ n=2\right)$$

The line integral of $\vec{F}$ along a curve from P to Q is defined to be

$$\underbrace{\int_C \vec{F}\cdot d\vec{r}}_{\text{notation}} = \int_a^b \vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t)\,dt.$$

This integral is indept of parametrization (from P to Q). When the parametrization is from Q to P, the integral get a "$-$" sign.

- When $\vec{F}$ is the force field, $\int_C \vec{F}\cdot d\vec{r}$ give the work done under $\vec{F}$ from P to Q along C.

e.g. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, $\vec{F} = z\hat{i} + xy\hat{j} - y^2\hat{k}$

$\vec{c}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}$, $t\in[0,1]$

$\vec{c}'(t) = 2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}$

$\updownarrow\ (x, y, z)\ \updownarrow$
$(t^2, t, \sqrt{t})$

$$\vec{F}(\vec{c}(t))\circ\vec{c}'(t) = (\sqrt{t}, t^3, -t^2)\circ\left(2t, 1, \frac{1}{2\sqrt{t}}\right)$$
$$= 2t^{3/2} + t^3 - t^2\frac{1}{2\sqrt{t}}$$
$$= \frac{3}{2}t^{3/2} + t^3$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(\frac{3}{2}t^{3/2} + t^3\right)dt = \cdots = \frac{17}{20} \quad \#$$

Step 1 Find a parametrization $\vec{c}(t)$ of C from P to Q.

Step 2 Calculate $\vec{c}'(t)$

Step 3 Find $\vec{F}(\vec{c}(t))\cdot\vec{c}'(t)$ and plug in
$$\int_a^b \vec{F}(\vec{c}(t))\circ\vec{c}'(t)\,dt.$$

eg Evaluate $\int_C -y\,dx + z\,dy + 2x\,dz$, C : $\cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$, $t\in[0, 2\pi]$.

This means $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (-y, z, 2x)$

In general,
$$\int_C M\,dx + N\,dy + P\,dz \equiv \int_C (M\hat{i} + N\hat{j} + P\hat{k})\cdot d\vec{r}.$$

$\vec{c}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

$\updownarrow\ (x, y, z)\ \updownarrow$
$(\cos t, \sin t, t)$

$$\vec{F}(\vec{c}(t))\cdot\vec{c}'(t) = (-\sin t, t, 2\cos t)\cdot(-\sin t, \cos t, 1)$$
$$= \sin^2 t + t\cos t + 2\cos t.$$

$$\therefore \int_C -y\,dx + z\,dy + 2x\,dz = \int_0^{2\pi} (\sin^2 t + t\cos t + 2\cos t)\,dt$$
$$= \pi \quad \#$$

Let $C_1, C_2, \dots, C_n$ be smooth curves. Suppose the endpt of C_{j-1} is equal to the starting point of C_j. Then we can put them together to form a piecewise smooth curve C, denoted by $C_1 + C_2 + \dots + C_n$. Define

$$\int_C f\,ds = \sum_{j=1}^{n} \int_{C_j} f\,ds,$$

$$\int_C \vec{F}\cdot d\vec{r} = \sum_{j=1}^{n} \int_{C_j} \vec{F}\cdot d\vec{r}.$$

e.g. Evaluate $\int_C (x - 3y^2 + z)\,ds$ when $C = C_1 + C_2$,
C_1: line segmet from $(0,0,0)$ to $(1,1,0)$
C_2: line segmet from $(1,1,0)$ to $(1,1,1)$

C_1 : $\vec{c}_1(t) = (0,0,0) + t((1,1,0) - (0,0,0))$
$= (t, t, 0), \quad t \in [0,1]$

$\therefore \vec{c}_1{}'(t) = (1,1,0)$

$|\vec{c}_1{}'(t)| = \sqrt{2}$

(x, y, z)
$\updownarrow \quad \updownarrow$
$(t, t, 0)$

$$\int_{C_1} (x - 3y^2 + z)\,ds = \int_0^1 (t - 3t^2 + 0)\sqrt{2}\,dt = -\frac{\sqrt{2}}{2}.$$

C_2 : $\vec{c}_2(t) = (1,1,0) + t((1,1,1) - (1,1,0))$
$= (1, 1, t), \quad t \in [0,1]$

$\vec{c}_2{}'(t) = (0,0,1)$,

$|\vec{c}_2{}'(t)| = 1$.

$$\int_{C_2} (x-3y^2+z)\,ds = \int_0^1 (1-3+t)\sqrt{1}\,dt$$

$(x, y, z) \updownarrow (1, 1, t)$

$$= -\frac{3}{2} \quad ,$$

$$\int_C (x-3y^2+z)\,ds = -\frac{\sqrt{2}}{2} - \frac{3}{2} . \quad \#$$