Week 11

March 29, 2023

2020 B Adv. Cal.

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An important v.f.

$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}.$$

It satisfies the Component Test but is NOT conservative. Let's see it.

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-1}{x^2+y^2} - (-y)\frac{2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2},$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2} - x\frac{(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ holds.

On the other hand, consider the circle

$\vec{c}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $t \in [0, 2\pi]$

$\vec{c}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$.

$$\oint_C M\,dx + N\,dy = \int_0^{2\pi} \frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t)\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = 2\pi$$

As $\oint_C \vec{F}\cdot d\vec{r} = 2\pi \neq 0$, $\vec{F}$ is not conservative.

This $\vec{F}$ is defined in $\mathbb{R}^2$ except at the origin.

View $\vec{F}$ as

$$\frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j} + 0\hat{k},$$

a v.f in $\mathbb{R}^3$ except at the z-axis. We now have

$$M = \frac{-y}{x^2+y^2},\ N = \frac{x}{x^2+y^2},\ P = 0.$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ already checked. $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y}$ also hold.

We still have $\vec{C}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\hat{k}$, $t \in [0, 2\pi]$

$$\oint_C \vec{F}\cdot d\vec{r} = 2\pi \neq 0.$$

So $\vec{F}$ is a v.f. in an open region in space satisfying the component test but is NOT conservative.

In Ex 8 I ask you to show the component Test is sufficient for $\vec{F}$ being conservative if $\vec{F}$ is smooth in the entire space $\mathbb{R}^n$.

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<u>Green's Thm</u> Let C be a simple, closed curve in the plane enclosing the region D. Suppose $\vec{F}$ is a smooth v.f. in D. Then

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA,$$ when C is oriented

anticlockwisely.

Pf in a special situation. Assume D can be described in 2 ways:

$$D = \{(x,y) : f_1(x) \le y \le f_2(x),\ a \le x \le b\}, \text{ and}$$
$$= \{(x,y) : g_1(y) \le x \le g_2(y),\ c \le y \le d\}.$$

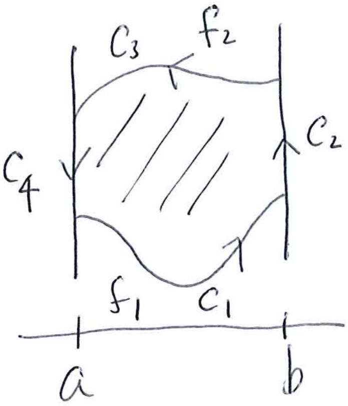
Then

$$\iint_D \frac{\partial M}{\partial y}\, dA = -\oint_C M\,dx, \quad (1)$$

$$\iint_D \frac{\partial N}{\partial x}\, dA = \oint_C N\,dy \quad (2)$$

Then $-(1)+(2) \Rightarrow$ Green's theorem.

Pf of (1)



$C = C_1 + C_2 + C_3 + C_4$,

$C_1 \quad \vec{C}_1(x) = x\,\hat{i} + f_1(x)\,\hat{j}, \quad x \in [a,b]$,

$$\therefore \int_{C_1} M\,dx = \int_a^b M(x, f_1(x))\,dx$$

$C_2 \quad \vec{C}_2(x) = b\,\hat{i} + [(1-t) f_1(b) + t f_2(b)], \quad t \in [0,1]$

$\vec{C}_2'(x) = 0\,\hat{i} + (-f_1(b) + f_2(b))\,\hat{j}$

$$\int_{C_2} M\,dx = \int_0^1 M(b, (1-t) f_1(b) + t f_2(b))\, 0\, dt = 0$$

$-C_3$: $\vec{r}(x) = x\,\hat{i} + f_2(x)\,\hat{j}, \quad x \in [a,b]$,

$\vec{r}'(x) = \hat{i} + f_2'(x)\,\hat{j}$.

$$\int_{C_3} M\,dx = -\int_{-C_3} M\,dx = -\int_a^b M(x, f_2(x))\, 1\, dx$$

$-C_4$: $\vec{r}(t) = a\,\hat{i} + ((1-t) f_1(a) + t f_2(a))\,\hat{j}, \quad t \in [0,1]$

$\vec{r}'(t) = 0\,\hat{i} + (-f_1(a) + f_2(a))\,\hat{j}$,

$$\int_{C_4} M\,dx = -\int_{-C_4} M(a, (1-t)f_1(a) + t f_2(a)) \cdot 0 \, dt = 0$$

$$\therefore \oint_C M\,dx = \sum_{j=1}^{4} \int_{C_j} M\,dx = \int_a^b M(x, f_1(x))\,dx - \int_a^b M(x, f_2(x))\,dx \, . \quad (3)$$

On the other hand,

$$\iint_D \frac{\partial M}{\partial y}\,dA = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial}{\partial y} M(x,y)\,dy\,dx$$

$$= \int_a^b M(x,y)\Big|_{y=f_1(x)}^{y=f_2(x)} dx$$

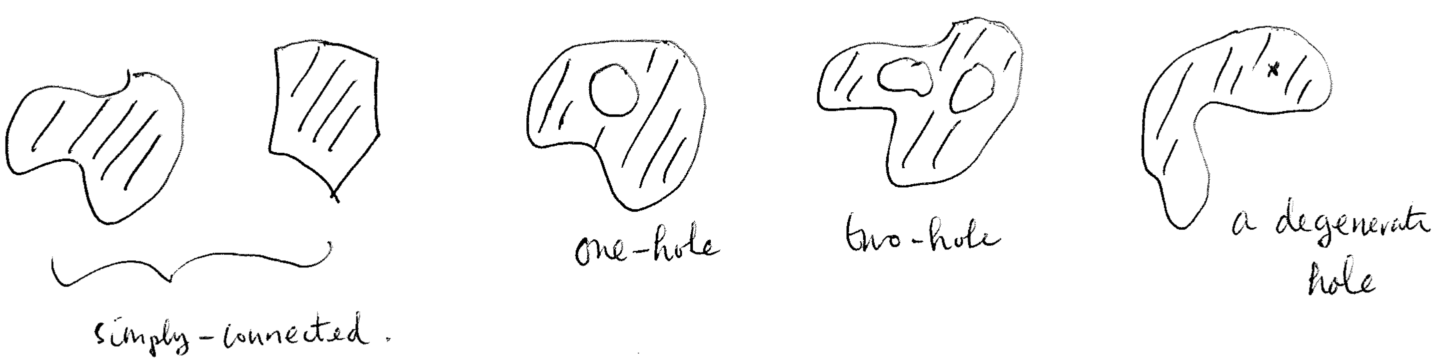
$$= \int_a^b M(x, f_2(x)) - M(x, f_1(x))\,dx \qquad (4)$$

Comparing (3) and (4), we see that (1) holds.

Similarly we can prove (2).

An important consequence of Green's theorem is this:

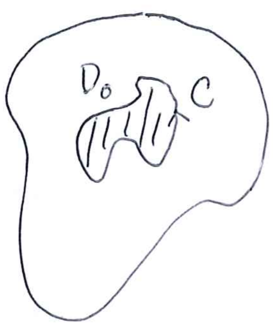
A region is simply-connected if it has no holes.



<u>Theorem</u> Let $\vec{F}$ be a smooth v.f. in a simply-connected region D. Then $\vec{F}$ is conservative iff it passes the Component Test.

Pf $\Rightarrow$) When $\vec{F}$ is conservative, the Component Test holds in any open region (not nec. simply-connected)

$\Leftarrow$) When D is simply-connected, let C be a simple closed curve in D enclosing D_0, $\vec{F}$ is well-defined in D_0, so

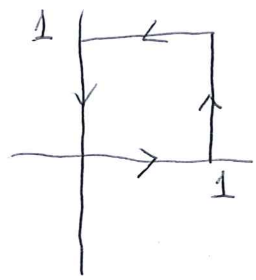


$$\oint_C M\,dx + N\,dy = \iint_{D_0} (N_x - M_y)\,dA = 0,$$

$\therefore$ $\vec{F}$ is conservative.

Next, we use Green's theorem to simplify calculations.

e.g. Evaluate $\oint_C xy\,dy - y^2\,dx$ when C is the square



Instead of doing 4 line integrals, we use

$$\oint_C xy\,dy - y^2\,dx = \iint_D \frac{\partial}{\partial x} xy - \frac{\partial}{\partial y}(-y^2)\,dA$$

$$= 3\iint_D y\,dA$$

$$= 3\int_0^1\int_0^1 y\,dy\,dx$$

$$= 3/2 \ \#$$

Third, the area formula. Take $N=\frac{1}{2}x$ and $M=-\frac{1}{2}y$ in Green's theorem.

$$\iint_D \frac{1}{2}-\left(-\frac{1}{2}\right)dA = \oint_C \frac{1}{2}x\,dy-\frac{1}{2}y\,dx$$

$$\therefore \quad \text{area of } D = \frac{1}{2}\oint_C x\,dy-y\,dx .$$

It is interesting to observe that the area can be found by just performing integration along its boundary.

We can rewrite Green's theorem in "flux form".

The original formula is

$$\iint_D (N_x-M_y)\,dA = \oint_C M\,dx+N\,dy$$

$$= \oint_C \vec{F}\cdot d\vec{r} \qquad (\text{the circulation of } \vec{F} \text{ around } C)$$

Change $N\to M$, $M\to -N$, we get

$$\iint_D (M_x+N_y)\,dA = \oint_C -N\,dx+M\,dy$$

$$= \oint_C \vec{F}\cdot\hat{n}\,ds \qquad (\text{the out flux of } \vec{F} \text{ across } C)$$

Note that for any C around a pt (x,y),

$$\frac{1}{|D|}\oint_C \vec{F}\cdot d\vec{r} = \frac{1}{|D|}\iint_D (N_x-M_y)\,dA \longrightarrow (N_x-M_y)(x,y)$$

as C shrinks to (x,y). It suggests the term $N_x - M_y$ is some kind of density for the circulation. Call it the curl of $\vec{F}$ at (x,y), denote it by curl $\vec{F}$. Hence Green's formula becomes

$$\iint_D \text{curl}\,\vec{F}\,dA = \oint_C M\,dx + N\,dy .$$

Similarly, the density for the flux is $M_x + N_y$, call it the divergence of $\vec{F}$ at (x,y), denote by div $\vec{F}$. We've

$$\iint_D \text{div}\,\vec{F}\,dA = \oint_C -N\,dx + M\,dy .$$

eg. Find the flux of $\vec{F} = 2e^{xy}\hat{i} + y^3\hat{j}$ out of the square at $x=\pm 1$, $y = \pm 1$.

$$\text{Flux} = \iint_D M_x + N_y\,dA = \iint_D (2ye^{xy} + 3y^2)\,dA$$

$$= \int_{-1}^{1}\int_{-1}^{1} (2ye^{xy} + 3y^2)\,dx\,dy$$

$$\vdots$$

$$= 4 \quad \#$$