March 22, 2023 

Week 10

2020 B Adv. Cal II

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Last week we discussed the line integral

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{c}(t))\cdot \vec{c}\,'(t)\,dt$$

When $\vec{F}$ is a force field, it stands for the work done by $\vec{F}$ along $\vec{c}$ (from $\vec{P}$ to $\vec{Q}$, say).

And yet there is another interpretation of this integral. When $\vec{F}$ is the velocity of the fluid, $\vec{F}\cdot\hat{t}$ measures the amount of fluid flowing along $\vec{c}$ in unit time. Hence

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{t}\,ds$$

is the amount of fluid flowing along $C$ in unit time. When $C$ is a simple closed curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, it is called the <u>circulation</u>.

e.g. Find the flow of $\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$ along $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$, $t \in [0, \pi/2]$.

$\vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$, $\quad \Updownarrow \begin{matrix} x, & y, & z \\ \cos t & \sin t & t \end{matrix} \updownarrow$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)$

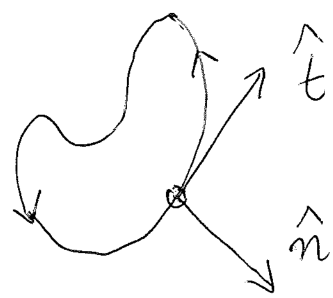
$= (\cos t\,\hat{i} + t\,\hat{j} + \sin t\,\hat{k})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k})$

$= (-\sin t\cos t + t\cos t + \sin t)$

$\therefore$ flow $= \int_0^{\pi/2} (-\sin t\cos t + t\cos t + \sin t)\,dt = \cdots = \frac{\pi - 1}{2}$ #

Let $C$ be a simple closed curve on the plane running in anticlockwise way. Its (unit) tangent is

$$\hat{t} = \frac{(x'(t), y'(t))}{\sqrt{x'^2(t)+y'^2(t)}}$$




and outer (unit) normal is

$$\hat{n} = \frac{(y'(t), -x'(t))}{\sqrt{x'^2(t)+y'^2(t)}}$$

The flux of $\vec{F}$ across $C$ is

$$\int_C \vec{F}\cdot\hat{n}\,ds$$

A formula:

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C M\,dy - N\,dx .$$

Pf:

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_a^b \vec{F}(\vec{c}(t))\cdot\frac{y'(t)\hat{i}-x'(t)\hat{j}}{\sqrt{x'^2+y'^2}}\sqrt{x'^2+y'^2}\,dt$$

$$= \int_a^b M(\vec{c}(t))\,y'(t) - N(\vec{c}(t))\,x'(t)\,dt$$

$$= \int_a^b M\,dy - N\,dx .$$

eg. Find the flux of $\vec{F} = (x-y)\hat{i} + x\hat{j}$ across the unit circle $x^2+y^2 = 1$.

Use the standard parametrization $x(t)\hat{i} + y(t)\hat{j} = \cos t\,\hat{i} + \sin t\,\hat{j}$, $t \in [0, 2\pi]$

$$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}, \quad x'(t) = -\sin t, \quad y'(t) = \cos t$$

$x, y$

$\cos t \quad \sin t$

$$\text{flux} = \int_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} (\cos t - \sin t)(\cos t) - \cos t(-\sin t)\,dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt$$

$$= \pi \ \#$$

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## Vector fields

An open region is simply a region removing its boundary points. More generally, an open region is defined to be a set $R$ in $\mathbb{R}^2, \mathbb{R}^3$ satisfying

ⓐ if $\vec{P} \in R$, then all points near $\vec{P}$ also belong to $R$.
(to exclude boundary pts)

ⓑ For any $\vec{P}, \vec{Q} \in R$, there is a smooth curve in $R$ connecting $\vec{P}$ and $\vec{Q}$.

Vector fields are very general objects.
We study special ones.

A vector field $\vec{F}$ in an open region is <u>conservative</u>

if $\oint_C \vec{F}\cdot d\vec{r} = 0$ for every closed path in the region.

A path is a piecewise closed curves.

A vector field $\vec{F}$ is <u>independent of path</u> if

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r},$$

whenever $C_1, C_2$ run from the same $\vec{P}$ to $\vec{Q}$.

A vector field $\vec{F}$ is a <u>gradient v.f.</u> if it is the gradient of some function, ie $\vec{F} = \nabla\Phi$, or

$$\frac{\partial\Phi}{\partial x} = M, \quad \frac{\partial\Phi}{\partial y} = N, \quad \frac{\partial\Phi}{\partial z} = P \quad (n=3)$$

$$\frac{\partial\Phi}{\partial x} = M, \quad \frac{\partial\Phi}{\partial y} = N \quad (n=2).$$

The function $\Phi$ is called a <u>potential</u> for $\vec{F}$. it is unique up to the addition of a constant.

<u>Main theorem</u> A smooth v.f. $\vec{F}$ is <u>conservative</u> iff it is indept of path iff it is gradient. Moreover, the potential $\Phi$ satisfies

$$\Phi(\vec{Q}) - \Phi(\vec{P}) = \int_C \vec{F}\cdot d\vec{r}, \text{ where } C \text{ is}$$

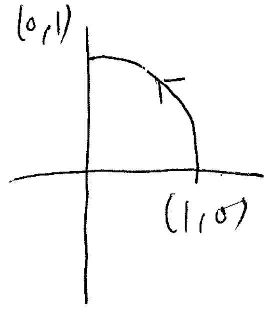
a path from $\vec{P}$ to $\vec{Q}$.

(proofs were present in class, see also Text. We omit it here.)

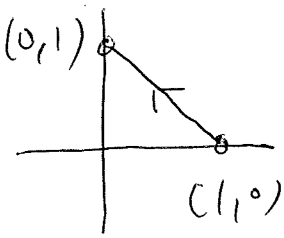
e.g. $\vec{F} = y^2\hat{i} + 2(x-1)y\hat{j}$

Consider three paths $C_1, C_2, C_3$ from $(1,0)$ to $(0,1)$

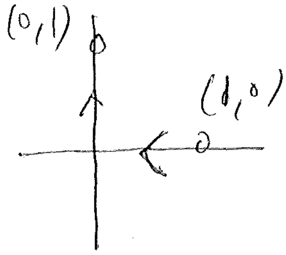
$C_1$




$\vec{C_1}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad t \in [0, \pi/2]$




$\vec{C_2}(t) = (1,0) + t\big((0,1)-(1,0)\big)$
$= (1-t, t), \quad t \in [0,1]$




$\vec{C_3} = \gamma_1 + \gamma_2,$
$\gamma_1(t) = (1-t)\,i \quad t \in [0,1]$
$\gamma_2(t) = t\,j \quad , t \in [0,1]$.

you can verify $\displaystyle\int_{C_1} \vec{F}\cdot dr = \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_3} \vec{F}\cdot d\vec{r}$.

In fact, $\vec{F}$ has a potential

$$\Phi(x,y) = (x-1)y^2.$$

$\dfrac{\partial \Phi}{\partial x} = y^2, \quad \dfrac{\partial \Phi}{\partial y} = 2(x-1)y$. yes.

Next, consider how to determine a v.f is gradient or not and find its potential if yes.

Component test

If $\vec{F}$ is gradient, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $(n=2)$, or

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial z} \quad (n=3).$$

Pf $(n=3)$ $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$. If gradient, then

$$M = \frac{\partial \Phi}{\partial x}, \; N = \frac{\partial \Phi}{\partial y}, \; P = \frac{\partial \Phi}{\partial z}. \quad \text{So}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \Phi}{\partial y \partial x}, \; \frac{\partial N}{\partial x} = \frac{\partial^2 \Phi}{\partial x \partial y} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = \frac{\partial^2 \Phi}{\partial z \partial x}, \; \frac{\partial P}{\partial x} = \frac{\partial^2 \Phi}{\partial x \partial z} \Rightarrow \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial z} = \frac{\partial^2 \Phi}{\partial z \partial y}, \; \frac{\partial P}{\partial y} = \frac{\partial^2 \Phi}{\partial y \partial z} \Rightarrow \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}.$$

e.g. Find a potential for

$$\vec{F} = -\frac{gMm}{r^3}(x, y, z)$$

(if any). Here $M = -\frac{gMm}{r^3}x$, $N = -\frac{gMm}{r^3}y$, $P = -\frac{gMm}{r^3}z$.

$$\frac{\partial M}{\partial y} = -gMmx\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-3/2} = -gMm x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2y)$$

$$= 3gMmxy/r^5$$

$$\frac{\partial N}{\partial x} = -gMmy\frac{\partial}{\partial x}(\quad)^{-3/2} = 3gMmyx/r^5, \quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Similarly, the others are fine. In fact, by observation,

$$\Phi = \frac{gMm}{r} = \frac{gMm}{(x^2+y^2+z^2)^{1/2}} .$$

eg. Find a potential for

$$\vec{F} = y^2 \hat{i} + 2(x-1)y \hat{j} .$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} y^2 = 2y, \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} 2(x-1)y = 2y . \quad \text{Pass the test.}$$

$$\frac{\partial \Phi}{\partial x} = y^2 \Rightarrow \Phi = xy^2 + g(y)$$

$$\frac{\partial \Phi}{\partial y} = 2xy + g'(y) = 2(x-1)y \quad (\text{should equal to})$$

$$= 2xy - 2y$$

$$\therefore \ g'(y) = -2y$$

$$g(y) = -y^2 + C$$

$$\therefore \ \Phi = xy^2 - y^2 + C ,$$

eg. Find a potential for

$$\vec{F} = (e^x \cos y + yz)\hat{i} + (xz - e^x \sin y)\hat{j} + (xy + z)\hat{k} .$$

$$\frac{\partial M}{\partial y} = -e^x \sin y + z, \quad \frac{\partial N}{\partial x} = z - e^x \sin y , \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} .$$

$$\frac{\partial M}{\partial z} = y, \quad \frac{\partial P}{\partial x} = y , \quad \therefore \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} .$$

$$\frac{\partial N}{\partial z} = x , \quad \frac{\partial P}{\partial y} = x \quad \therefore \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} .$$

$\vec{F}$ passes the Component Test.

$$\frac{\partial \Phi}{\partial x} = M = e^x \cos y + yz$$

$$\therefore \quad \Phi = e^x \cos y + yzx + g(y,z)$$

$$\frac{\partial \Phi}{\partial y} = -e^x \sin y + xz + \frac{\partial g}{\partial y}$$

$$= xz - e^x \sin y \quad \text{(should eql to)}$$

$$\therefore \frac{\partial g}{\partial y} = 0, \quad \text{ie} \quad g(y,z) = h(z).$$

$$\frac{\partial \Phi}{\partial z} = xy + h'(z)$$

$$= xy + z \quad \text{(should eql to)}$$

$$\therefore \quad h'(z) = z, \quad h(z) = \frac{z^2}{2} + C.$$

Potential is

$$\Phi = e^x \cos y + xyz + \frac{z^2}{2} + C \quad \#$$