Quiz 2

Answer all questions. Make your presentation clear and clean. Each question carries 10 marks.

1. Let D be the region bounded by $xy = 1, xy = 3, y = x$ and $y = 5x$ in the first quadrant. Find its area by using a change of variables to transform D into a rectangle.

Let $u = xy \in [1,3]$, $v = y/x \in [1,5]$. then $x = u^{\frac{1}{2}}v^{-\frac{1}{2}}$, $y = u^{\frac{1}{2}}v^{\frac{1}{2}}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}u^{-\frac{3}{2}} \\ \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix} = \frac{1}{2}\frac{1}{v}. \quad \therefore \text{area} = \int_1^3\int_1^5 \frac{1}{2}\frac{1}{v}\,dv\,du = \log 5.$$

2. Let $C = C_1 + C_2$ where C_1 is the line segment from $(0, 0, 0)$ to $(-1, 2, 0)$ and C_2 is the line segment from $(-1, 2, 0)$ to $(0, 2, 1/2)$. Evaluate the line integral

$$\int_C y \sin \pi z \, ds .$$

$\vec{C}_1(t) = (0,0,0) + t((-1,2,0) - (0,0,0))$
$= (-t, 2t, 0)$, $t \in [0,1]$

$\vec{C}_1'(t) = (-1,2,0)$, $|\vec{C}_1'(t)| = \sqrt{5}$

$\int_{C_1} y \sin \pi z\, ds = \int_0^1 2t \sin \pi 0 \sqrt{5}\, dt = 0.$

$\vec{C}_2(t) = (-1,2,0) + t((0,2,\frac{1}{2}) - (-1,2,0))$
$= (-1+t, 2, t/2)$

$\vec{C}_2'(t) = (1, 0, 1/2)$, $|\vec{C}_2'(t)| = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$

$\int_{C_2} y \sin \pi z\, ds = \int_0^1 2 \sin \frac{\pi}{2} t \frac{\sqrt{5}}{2} dt$
$= \frac{2\sqrt{5}}{\pi}$

$\therefore \int_C y \sin \pi z\, ds = \int_{C_1} y \sin \pi z\, ds + \int_{C_2} y \sin \pi z\, ds$
$= 0 + \frac{2\sqrt{5}}{\pi} = \frac{2\sqrt{5}}{\pi}.$

3. Let $\mathbf{F} = 7y\mathbf{i} - x\mathbf{j}$. Find (a) the circulation of $\mathbf{F}$ around the unit circle $x^2 + y^2 = 1$ oriented in the anticlockwise way and (b) the outward flux of $\mathbf{F}$ across the same oriented circle.

$\vec{C}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $\vec{C}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$,

(a) Circulation $= \oint_C M\,dx + N\,dy = \int_0^{2\pi} 7\sin t(-\sin t) - \cos t(\cos t)\,dt$
$= \int_0^{2\pi} -7\sin^2 t - \cos^2 t\,dt = \cdots = -8\pi.$

(b) flux $= \oint_C M\,dy - N\,dx = \int_0^{2\pi} 7\sin t(\cos t) - \cos t(-\sin t)\,dt$
$= 8\int_0^{2\pi} \sin t \cos t\,dt = 4\int_0^{2\pi} \sin 2t\,dt$
$= 0$ #