Midterm Examination

Answer all nine questions. You should justify your answer by showing all steps.

1. (10 points) Let D be the region bounded by $3x = y^2$ and $x = 1$. Evaluate
$$\iint_D (x - 3y)\, dA \ .$$

2. (10 points) Evaluate
$$\int_0^1 \int_{2x}^2 y^2 e^{xy}\, dydx \ .$$

3. (10 points) Find the area of the region D which is the region lying inside the circle $r = 3\cos\theta$ but outside the cardioid $r = 1 + \cos\theta$.

4. (10 points) Show that
$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \ .$$

5. (15 points) Let D be a plane region unchanged under the map $(x, y) \mapsto (-x, -y)$.

   (a) For a continuous function in D satisfying $f(-x, -y) = -f(x, y)$, show that
$$\iint_D f(x, y)\, dA = 0 \ .$$

   (b) Deduce that the centroid of D is the origin.

6. (10 points) Find the volume of the region Ω in space which is bounded between the surfaces of $z = x^2$ and $z = 6 - x^2 - 2y^2$.

7. (10 points) Let S be the solid bounded above by the sphere $x^2 + y^2 + z^2 \le 4$ and below by the plane $z = \sqrt{2}$. Suppose its density is $\delta(x, y, z) = z$. Find its moment of inertia I_z using spherical coordinates.

8. (15 points) The plane $x + 2y + 3z = 6$ and the three coordinate planes form a tetrahedron T. Express the integral
$$\iiint_T f(x, y, z)\, dV$$
as an iterated integral in (a) $dzdxdy$ and in (c) in cylindrical coordinates.

9. (10 points) Express the iterated integral
$$\int_{-\pi/2}^{\pi/2} \int_0^3 \int_0^{\sqrt{16-r^2}} (1 + z^2) r^4 \sin^2\theta \cos\theta\, dzdrd\theta$$
as an iterated integral in (a) spherical coordinates and (b) in $dxdzdy$. You need not evaluate it.

Midterm Exam Solution

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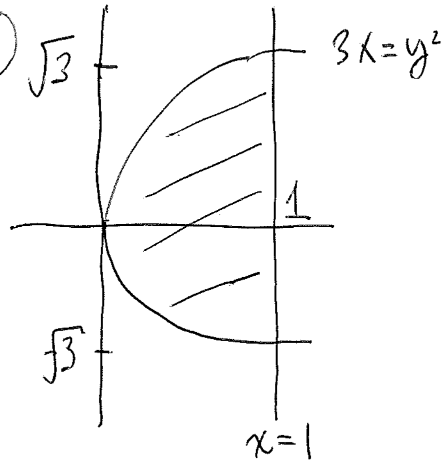
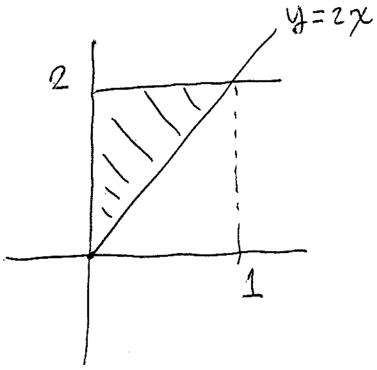
(1)



$$\iint_D (x-3y)\,dA = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{y^2/3}^{1} (x-3y)\,dx\,dy$$

$$= \cdots = \frac{4\sqrt{3}}{5} \quad \#$$

(2)



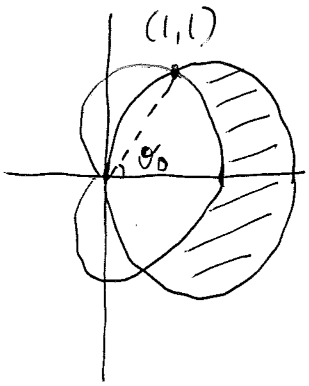
Change order of integration

$$\int_0^1 \int_{2x}^{2} y^2 e^{xy}\,dy\,dx = \int_0^2 \int_0^{y/2} y^2 e^{xy}\,dx\,dy$$

$$= \int_0^2 y e^{xy} \Big|_{x=0}^{x=y/2} dy = \int_0^2 \left(y e^{y^2/2} - y\right) dy$$

$$= \cdots = e^2 - 3 \quad \#$$

(3)



$r = 3\cos\theta$ and $r = 1+\cos\theta$ intersects at $3\cos\theta = 1+\cos\theta$,

$\cos\theta = 1/2$, $\theta_0 = \pi/3$

$$\therefore \text{area} = 2\int_0^{\pi/3} \int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta$$

$$= \pi \quad \#$$

(4) Let $I = \int_0^\infty e^{-x^2} dx$.

$$I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = \int_0^\infty \int_0^\infty e^{-x^2-y^2} dx\,dy$$

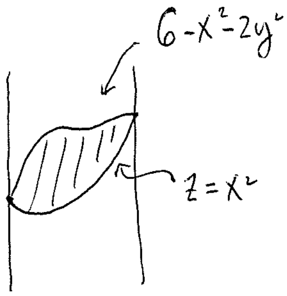
$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta = \frac{-1}{2} \int_0^{\pi/2} e^{-r^2}\Big|_0^\infty d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \pi/4.$$

$$\therefore I = \frac{\sqrt{\pi}}{2} \quad \#$$

(5) See Solution 5.

(6)



A cross section

the surfaces intersect at
$x^2 = 6 - x^2 - 2y^2$, $x^2 + y^2 = 3$.

$\therefore \Omega$: $x^2 \le z \le 6 - x^2 - 2y^2$
$(x,y) \in D_{\sqrt{3}}$ (disk of radius $\sqrt{3}$).

$$\text{vol of } \Omega = \iint_{D_{\sqrt{3}}} \int_{x^2}^{6-x^2-2y^2} 1\, dz\, dA(x,y)$$

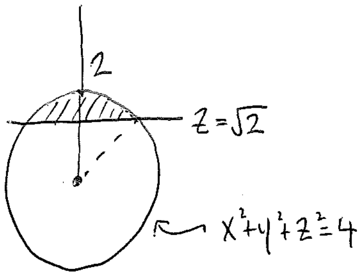
$$= \iint_{D_{\sqrt{3}}} (6 - x^2 - 2y^2 - x^2)\, dA(x,y)$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} (6 - 2r^2) r\,dr\,d\theta$$

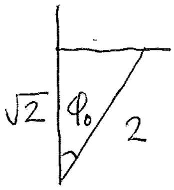
$\vdots$

$$= 9\pi \quad \#$$

(7)



a cross section



$\cos\varphi_0 = \frac{\sqrt{2}}{2}$

$\varphi_0 = \pi/4$

$$I_z = \iiint_{\Omega} (x^2+y^2)\,\delta\, dV(x,y,z)$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_{\sqrt{2}/\cos\varphi}^{2} \left[(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2\right] \underset{\wedge}{}\rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta \qquad \rho\cos\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_{\sqrt{2}/\cos\varphi}^{2} \rho^5 \sin^3\varphi\cos\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{6}\left(64-\frac{8}{\cos^6\varphi}\right)\sin^3\varphi\cos\varphi\, d\varphi\, d\theta$$

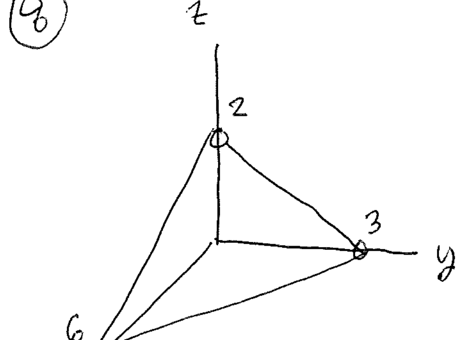
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{32}{3}\sin^3\varphi\cos\varphi\, d\varphi\, d\theta$$

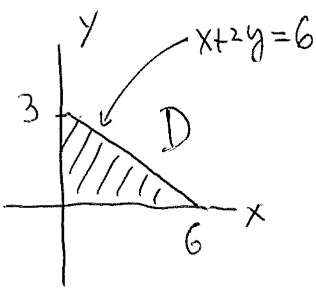
$$- \int_0^{2\pi}\int_0^{\pi/4} \frac{4}{3}\frac{\sin^3\varphi}{\cos^5\varphi}\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \frac{8}{3}\sin^4\varphi\Big|_0^{\pi/4} d\theta - \int_0^{2\pi} \frac{4}{3}\tan^3\varphi\, d(\tan\varphi)\, d\theta$$

$$= \frac{4\pi}{3} - \frac{2\pi}{3} = \frac{2\pi}{3} \quad \#$$

(8)





as a graph over (x,y)-plane

$T.\quad 0 \le z \le \frac{1}{3}(6-x-2y)$

$(x,y) \in$ a triangle (see figure) D

$$\therefore \iiint_T f\, dV = \iint_D \int_0^{\frac{1}{3}(6-x-2y)} f(x,y,z)\, dz\, dA(x,y)$$

$$= \int_0^3 \int_0^{6-2y} \int_0^{\frac{1}{3}(6-x-2y)} f(x,y,z)\, dz\, dx\, dy \,.$$

In cylindrical coor, $x+2y=6$ becomes $r = 6/(\cos\theta + 2\sin\theta)$

$$\iiint_T f\,dV = \int_0^{\pi/2} \int_0^{6/(\cos\theta+2\sin\theta)} \int_0^{\frac{1}{3}(6 - r\cos\theta - 2r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, dz\, r\,dr\,d\theta .$$

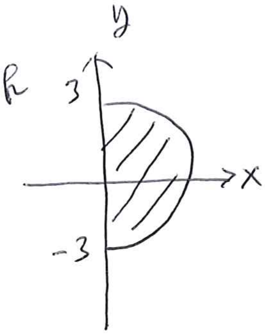
(9) By writing

$$\int_{-\pi/2}^{\pi/2} \int_0^3 \int_0^{\sqrt{16-r^2}} (1+z^2)\, r^4 \sin^2\theta \cos\theta \, dz\,dr\,d\theta \quad \text{as}$$

$$\int_{-\pi/2}^{\pi/2} \int_0^3 \int_0^{\sqrt{16-r^2}} (1+z^2)(r\sin\theta)^2 (r\cos\theta)\, r\,dz\,dr\,d\theta ,$$

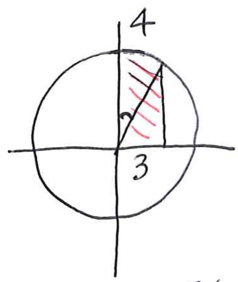
we see that

$$f(x,y,z) = (1+z^2)\, x y^2 .$$



Ω is the sphere $x^2+y^2+z^2=16$ over the half disk in the (x,y)-plane, ie,

$$\Omega = \Omega_1 \cup \Omega_2 ,$$



Cross section

Ω_1 : $0 \le \rho \le 4$
$\quad 0 \le \varphi \le \varphi_0$
$\quad -\pi/2 \le \theta \le \pi/2$.

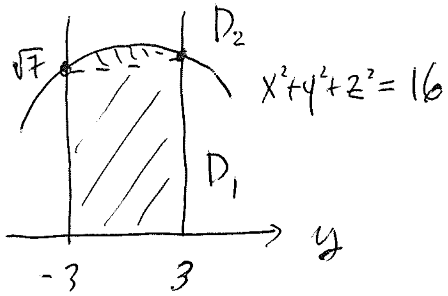
$\sin\varphi_0 = 3/4$, $\varphi_0 = \sin^{-1}\frac{3}{4}$.

Ω_2 : $0 \le \rho \le 3/\sin\varphi \quad (\because x^2+y^2 \le 9)$
$\quad \varphi_0 \le \varphi \le \pi/2$
$\quad -\pi/2 \le \theta \le \pi/2$.

$\therefore$ Integral $$= \int_{-\pi/2}^{\pi/2} \int_0^{\varphi_0} \int_0^4 (1+\rho^2\cos^2\varphi) \sin^4\varphi \cos\theta \sin^2\theta \, \rho^5 \, d\rho\, d\varphi\, d\theta$$

$$+ \int_{-\pi/2}^{\pi/2} \int_{\varphi_0}^{\pi/2} \int_0^{3/\sin\varphi} (\cdots)\, d\rho\, d\varphi\, d\theta .$$

can be viewed as a graph over the region in y-z plane.



Over D_1, it is part of the cylinder $x^2+y^2=9$, ie

$$0 \le x \le \sqrt{9-y^2}$$
$$-3 \le y \le 3$$
$$0 \le z \le \sqrt{7}.$$

Over D_2, it is part of the sphere $x^2+y^2+z^2=16$, ie

$$0 \le x \le \sqrt{16-y^2-z^2}$$
$$-3 \le y \le 3$$
$$\sqrt{7} \le z \le \sqrt{16-y^2}$$

$$\text{Integral} = \iiint_{\Omega} (1+z^2)xy^2\,dV$$

$$= \iint_{D_1} \int_0^{\sqrt{9-y^2}} (1+z^2)xy^2\,dx\,dA(y,z) + \iint_{D_2} \int_0^{\sqrt{16-y^2-z^2}} (1+z^2)xy^2\,dx\,dA(y,z)$$

$$= \int_{-3}^{3}\int_0^{\sqrt{7}}\int_0^{\sqrt{9-y^2}} (1+z^2)xy^2\,dx\,dz\,dy + \int_{-3}^{3}\int_{\sqrt{7}}^{\sqrt{16-y^2}}\int_0^{\sqrt{16-y^2-z^2}} (1+z^2)xy^2\,dx\,dz\,dy \quad \#$$