# Exercises 16.2

## Vector Fields

Find the gradient fields of the functions in Exercises 1–4.

1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
2. $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$
3. $g(x, y, z) = e^z - \ln(x^2 + y^2)$
4. $g(x, y, z) = xy + yz + xz$
5. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the property that $\mathbf{F}$ points toward the origin with magnitude inversely proportional to the square of the distance from $(x, y)$ to the origin. (The field is not defined at $(0, 0)$.)
6. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the properties that $\mathbf{F} = \mathbf{0}$ at $(0, 0)$ and that at any other point $(a, b)$, $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.

## Line Integrals of Vector Fields

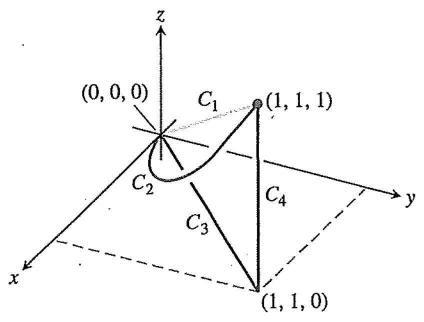
In Exercises 7–12, find the line integrals of $\mathbf{F}$ from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths in the accompanying figure.

a. The straight-line path $C_1$: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $\quad 0 \le t \le 1$

b. The curved path $C_2$: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $\quad 0 \le t \le 1$

c. The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the segment from $(1, 1, 0)$ to $(1, 1, 1)$

7. $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$
8. $\mathbf{F} = [1/(x^2 + 1)]\mathbf{j}$
9. $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$
10. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
11. $\mathbf{F} = (3x^2 - 3x)\mathbf{i} + 3z\mathbf{j} + \mathbf{k}$
12. $\mathbf{F} = (y + z)\mathbf{i} + (z + x)\mathbf{j} + (x + y)\mathbf{k}$



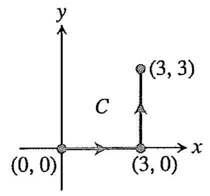

## Line Integrals with Respect to x, y, and z

In Exercises 13–16, find the line integrals along the given path $C$.

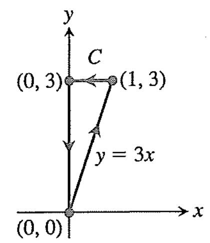
13. $\displaystyle\int_C (x - y)\,dx$, where $C$: $x = t, y = 2t + 1$, for $0 \le t \le 3$

14. $\displaystyle\int_C \frac{x}{y}\,dy$, where $C$: $x = t, y = t^2$, for $1 \le t \le 2$

15. $\displaystyle\int_C (x^2 + y^2)\,dy$, where $C$ is given in the accompanying figure




16. $\displaystyle\int_C \sqrt{x + y}\,dx$, where $C$ is given in the accompanying figure




17. Along the curve $\mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 1$, evaluate each of the following integrals.

a. $\displaystyle\int_C (x + y - z)\,dx$ $\qquad$ b. $\displaystyle\int_C (x + y - z)\,dy$

c. $\displaystyle\int_C (x + y - z)\,dz$

18. Along the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k}$, $0 \le t \le \pi$, evaluate each of the following integrals.

a. $\displaystyle\int_C xz\,dx$ $\qquad$ b. $\displaystyle\int_C xz\,dy$ $\qquad$ c. $\displaystyle\int_C xyz\,dz$

## Work

In Exercises 19–22, find the work done by $\mathbf{F}$ over the curve in the direction of increasing $t$.

19. $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$
$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$

20. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$
$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \le t \le 2\pi$

21. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 2\pi$

22. $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k}$
$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \le t \le 2\pi$

## Line Integrals in the Plane

23. Evaluate $\int_C xy\,dx + (x + y)\,dy$ along the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

24. Evaluate $\int_C (x - y)\,dx + (x + y)\,dy$ counterclockwise around the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

25. Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ for the vector field $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ along the curve $x = y^2$ from $(4, 2)$ to $(1, -1)$.

26. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ counterclockwise along the unit circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$.

### Work, Circulation, and Flux in the Plane

**27. Work** Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$ over the straight line from (1, 1) to (2, 3).

**28. Work** Find the work done by the gradient of $f(x, y) = (x + y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from (2, 0) to itself.

**29. Circulation and flux** Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

a. The circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

b. The ellipse $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

**30. Flux across a circle** Find the flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

In Exercises 31–34, find the circulation and flux of the field $\mathbf{F}$ around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \le t \le \pi$, followed by the line segment $\mathbf{r}_2(t) = t\mathbf{i}$, $-a \le t \le a$.

**31.** $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

**32.** $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$

**33.** $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

**34.** $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$

**35. Flow integrals** Find the flow of the velocity field $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$ along each of the following paths from (1, 0) to (−1, 0) in the $xy$-plane.

a. The upper half of the circle $x^2 + y^2 = 1$

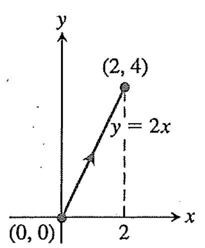
b. The line segment from (1, 0) to (−1, 0)

c. The line segment from (1, 0) to (0, −1) followed by the line segment from (0, −1) to (−1, 0)

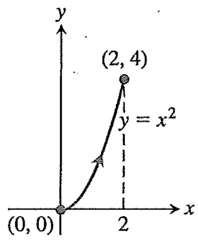
**36. Flux across a triangle** Find the flux of the field $\mathbf{F}$ in Exercise 35 outward across the triangle with vertices (1, 0), (0, 1), (−1, 0).

**37.** Find the flow of the velocity field $\mathbf{F} = y^2\mathbf{i} + 2xy\mathbf{j}$ along each of the following paths from (0, 0) to (2, 4).

a.



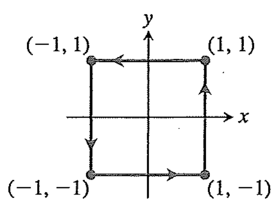

b.



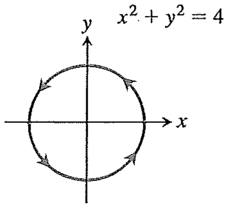

c. Use any path from (0, 0) to (2, 4) different from parts (a) and (b).

**38.** Find the circulation of the field $\mathbf{F} = y\mathbf{i} + (x + 2y)\mathbf{j}$ around each of the following closed paths.

a.




b.




c. Use any closed path different from parts (a) and (b).

### Vector Fields in the Plane

**39. Spin field** Draw the spin field

$$\mathbf{F} = -\frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$$

(see Figure 16.12) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 4$.

**40. Radial field** Draw the radial field

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$

(see Figure 16.11) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 1$.

**41. A field of tangent vectors**

a. Find a field $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the $xy$-plane with the property that at any point $(a, b) \ne (0, 0)$, $\mathbf{G}$ is a vector of magnitude $\sqrt{a^2 + b^2}$ tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the counterclockwise direction. (The field is undefined at (0, 0).)

b. How is $\mathbf{G}$ related to the spin field $\mathbf{F}$ in Figure 16.12?

**42. A field of tangent vectors**

a. Find a field $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the $xy$-plane with the property that at any point $(a, b) \ne (0, 0)$, $\mathbf{G}$ is a unit vector tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the clockwise direction.

b. How is $\mathbf{G}$ related to the spin field $\mathbf{F}$ in Figure 16.12?

**43. Unit vectors pointing toward the origin** Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the $xy$-plane with the property that at each point $(x, y) \ne (0, 0)$, $\mathbf{F}$ is a unit vector pointing toward the origin. (The field is undefined at (0, 0).)

**44. Two "central" fields** Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the $xy$-plane with the property that at each point $(x, y) \ne (0, 0)$, $\mathbf{F}$ points toward the origin and $|\mathbf{F}|$ is **(a)** the distance from $(x, y)$ to the origin, **(b)** inversely proportional to the distance from $(x, y)$ to the origin. (The field is undefined at (0, 0).)

**45. Work and area** Suppose that $f(t)$ is differentiable and positive for $a \le t \le b$. Let $C$ be the path $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}$, $a \le t \le b$, and $\mathbf{F} = y\mathbf{i}$. Is there any relation between the value of the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and the area of the region bounded by the $t$-axis, the graph of $f$, and the lines $t = a$ and $t = b$? Give reasons for your answer.

**46. Work done by a radial force with constant magnitude** A particle moves along the smooth curve $y = f(x)$ from $(a, f(a))$ to

**EXAMPLE 6** Show that $y\,dx + x\,dy + 4\,dz$ is exact and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

over any path from (1, 1, 1) to (2, 3, −1).

**Solution** We let $M = y, N = x, P = 4$ and apply the Test for Exactness:

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}.$$

These equalities tell us that $y\,dx + x\,dy + 4\,dz$ is exact, so

$$y\,dx + x\,dy + 4\,dz = df$$

for some function $f$, and the integral's value is $f(2, 3, -1) - f(1, 1, 1)$.

We find $f$ up to a constant by integrating the equations

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 4. \tag{4}$$

From the first equation we get

$$f(x, y, z) = xy + g(y, z).$$

The second equation tells us that

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x, \quad \text{or} \quad \frac{\partial g}{\partial y} = 0.$$

Hence, $g$ is a function of $z$ alone, and

$$f(x, y, z) = xy + h(z).$$

The third of Equations (4) tells us that

$$\frac{\partial f}{\partial z} = 0 + \frac{dh}{dz} = 4, \quad \text{or} \quad h(z) = 4z + C.$$

Therefore,

$$f(x, y, z) = xy + 4z + C.$$

The value of the line integral is independent of the path taken from (1, 1, 1) to (2, 3, −1), and equals

$$f(2, 3, -1) - f(1, 1, 1) = 2 + C - (5 + C) = -3.$$

■

## Exercises 16.3

### Testing for Conservative Fields

Which fields in Exercises 1–6 are conservative, and which are not?

1. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
2. $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$
3. $\mathbf{F} = y\mathbf{i} + (x + z)\mathbf{j} - y\mathbf{k}$
4. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$
5. $\mathbf{F} = (z + y)\mathbf{i} + z\mathbf{j} + (y + x)\mathbf{k}$
6. $\mathbf{F} = (e^x \cos y)\mathbf{i} - (e^x \sin y)\mathbf{j} + z\mathbf{k}$

### Finding Potential Functions

In Exercises 7–12, find a potential function $f$ for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$
8. $\mathbf{F} = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$
9. $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$
10. $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$
11. $\mathbf{F} = (\ln x + \sec^2(x + y))\mathbf{i} + \left(\sec^2(x + y) + \dfrac{y}{y^2 + z^2}\right)\mathbf{j} + \dfrac{z}{y^2 + z^2}\mathbf{k}$
12. $\mathbf{F} = \dfrac{y}{1 + x^2y^2}\mathbf{i} + \left(\dfrac{x}{1 + x^2y^2} + \dfrac{z}{\sqrt{1 - y^2z^2}}\right)\mathbf{j} + \left(\dfrac{y}{\sqrt{1 - y^2z^2}} + \dfrac{1}{z}\right)\mathbf{k}$

### Exact Differential Forms

In Exercises 13–17, show that the differential forms in the integrals are exact. Then evaluate the integrals.

13. $\int_{(0,0,0)}^{(2,3,-6)} 2x\,dx + 2y\,dy + 2z\,dz$

14. $\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$

15. $\int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$

16. $\int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1+z^2}\,dz$

17. $\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\,dx + \cos y \sin x\,dy + dz$

### Finding Potential Functions to Evaluate Line Integrals

Although they are not defined on all of space $R^3$, the fields associated with Exercises 18–22 are conservative. Find a potential function for each field and evaluate the integrals as in Example 6.

18. $\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}\,dz$

19. $\int_{(1,1,1)}^{(1,2,3)} 3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln y\,dz$

20. $\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

21. $\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}\,dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}\,dz$

22. $\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}$

### Applications and Examples

23. **Revisiting Example 6** Evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

from Example 6 by finding parametric equations for the line segment from (1, 1, 1) to (2, 3, −1) and evaluating the line integral of $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + 4\mathbf{k}$ along the segment. Since **F** is conservative, the integral is independent of the path.

24. Evaluate

$$\int_C x^2\,dx + yz\,dy + (y^2/2)\,dz$$

along the line segment $C$ joining (0, 0, 0) to (0, 3, 4).

**Independence of path** Show that the values of the integrals in Exercises 25 and 26 do not depend on the path taken from $A$ to $B$.

25. $\int_A^B z^2\,dx + 2y\,dy + 2xz\,dz$

26. $\int_A^B \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}}$

In Exercises 27 and 28, find a potential function for **F**.

27. $\mathbf{F} = \frac{2x}{y}\mathbf{i} + \left(\frac{1 - x^2}{y^2}\right)\mathbf{j}, \quad \{(x, y): y > 0\}$

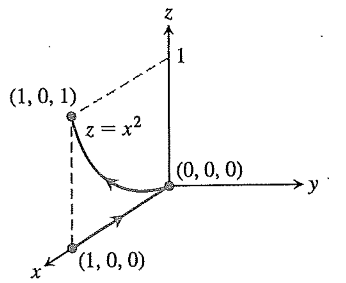
28. $\mathbf{F} = (e^x \ln y)\mathbf{i} + \left(\frac{e^x}{y} + \sin z\right)\mathbf{j} + (y\cos z)\mathbf{k}$

29. **Work along different paths** Find the work done by $\mathbf{F} = (x^2 + y)\mathbf{i} + (y^2 + x)\mathbf{j} + ze^z\mathbf{k}$ over the following paths from (1, 0, 0) to (1, 0, 1).

a. The line segment $x = 1, y = 0, 0 \le z \le 1$

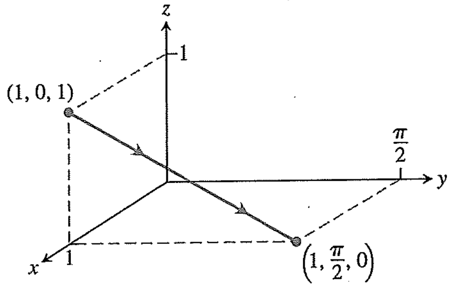
b. The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/2\pi)\mathbf{k}, 0 \le t \le 2\pi$

c. The $x$-axis from (1, 0, 0) to (0, 0, 0) followed by the parabola $z = x^2, y = 0$ from (0, 0, 0) to (1, 0, 1)

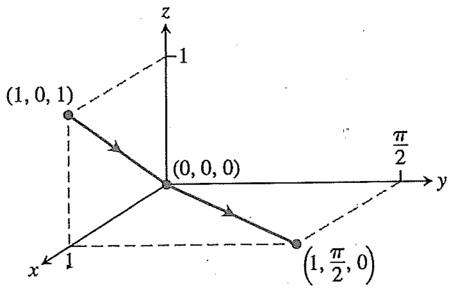



30. **Work along different paths** Find the work done by $\mathbf{F} = e^{yz}\mathbf{i} + (xze^{yz} + z\cos y)\mathbf{j} + (xye^{yz} + \sin y)\mathbf{k}$ over the following paths from (1, 0, 1) to $(1, \pi/2, 0)$.

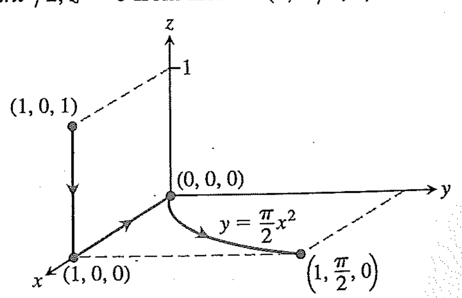
a. The line segment $x = 1, y = \pi t/2, z = 1 - t, 0 \le t \le 1$




b. The line segment from (1, 0, 1) to the origin followed by the line segment from the origin to $(1, \pi/2, 0)$




c. The line segment from (1, 0, 1) to (1, 0, 0), followed by the $x$-axis from (1, 0, 0) to the origin, followed by the parabola $y = \pi x^2/2, z = 0$ from there to $(1, \pi/2, 0)$



**31. Evaluating a work integral two ways** Let $\mathbf{F} = \nabla(x^3y^2)$ and let $C$ be the path in the $xy$-plane from $(-1, 1)$ to $(1, 1)$ that consists of the line segment from $(-1, 1)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(1, 1)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.

a. Find parametrizations for the segments that make up $C$ and evaluate the integral.

b. Use $f(x, y) = x^3y^2$ as a potential function for $\mathbf{F}$.

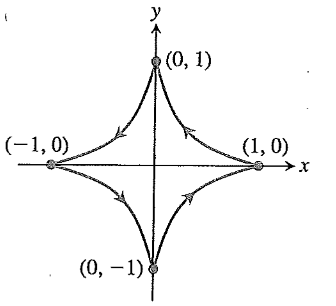
**32. Integral along different paths** Evaluate the line integral $\int_C 2x \cos y\, dx - x^2 \sin y\, dy$ along the following paths $C$ in the $xy$-plane.

a. The parabola $y = (x - 1)^2$ from $(1, 0)$ to $(0, 1)$

b. The line segment from $(-1, \pi)$ to $(1, 0)$

c. The $x$-axis from $(-1, 0)$ to $(1, 0)$

d. The astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}$, $0 \le t \le 2\pi$, counterclockwise from $(1, 0)$ back to $(1, 0)$




**33. a. Exact differential form** How are the constants $a$, $b$, and $c$ related if the following differential form is exact?

$$(ay^2 + 2czx)\, dx + y(bx + cz)\, dy + (ay^2 + cx^2)\, dz$$

**b. Gradient field** For what values of $b$ and $c$ will

$$\mathbf{F} = (y^2 + 2czx)\mathbf{i} + y(bx + cz)\mathbf{j} + (y^2 + cx^2)\mathbf{k}$$

be a gradient field?

**34. Gradient of a line integral** Suppose that $\mathbf{F} = \nabla f$ is a conservative vector field and

$$g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r}.$$

Show that $\nabla g = \mathbf{F}$.

**35. Path of least work** You have been asked to find the path along which a force field $\mathbf{F}$ will perform the least work in moving a particle between two locations. A quick calculation on your part shows $\mathbf{F}$ to be conservative. How should you respond? Give reasons for your answer.

**36. A revealing experiment** By experiment, you find that a force field $\mathbf{F}$ performs only half as much work in moving an object along path $C_1$ from $A$ to $B$ as it does in moving the object along path $C_2$ from $A$ to $B$. What can you conclude about $\mathbf{F}$? Give reasons for your answer.

**37. Work by a constant force** Show that the work done by a constant force field $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in moving a particle along any path from $A$ to $B$ is $W = \mathbf{F} \cdot \overrightarrow{AB}$.

**38. Gravitational field**

a. Find a potential function for the gravitational field

$$\mathbf{F} = -GmM\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

($G$, $m$, and $M$ are constants).

b. Let $P_1$ and $P_2$ be points at distance $s_1$ and $s_2$ from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from $P_1$ to $P_2$ is

$$GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right).$$

# 16.4 Green's Theorem in the Plane

If $\mathbf{F}$ is a conservative field, then we know $\mathbf{F} = \nabla f$ for a differentiable function $f$, and we can calculate the line integral of $\mathbf{F}$ over any path $C$ joining point $A$ to $B$ as $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$. In this section we derive a method for computing a work or flux integral over a *closed* curve $C$ in the plane when the field $\mathbf{F}$ is *not* conservative. This method comes from Green's Theorem, which allows us to convert the line integral into a double integral over the region enclosed by $C$.

The discussion is given in terms of velocity fields of fluid flows (a fluid is a liquid or a gas) because they are easy to visualize. However, Green's Theorem applies to any vector field, independent of any particular interpretation of the field, provided the assumptions of the theorem are satisfied. We introduce two new ideas for Green's Theorem: *circulation density* around an axis perpendicular to the plane and *divergence* (or *flux density*).

## Spin Around an Axis: The k-Component of Curl

Suppose that $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is the velocity field of a fluid flowing in the plane and that the first partial derivatives of $M$ and $N$ are continuous at each point of a region $R$. Let $(x, y)$ be a point in $R$ and let $A$ be a small rectangle with one corner at $(x, y)$ that, along with its interior, lies entirely in $R$. The sides of the rectangle, parallel to the coordinate axes, have lengths of $\Delta x$ and $\Delta y$. Assume that the components $M$ and $N$ do not