Solution 9

16.4 no 28. Solution. Call the arch of the cycloid running from $t = 0$ to 2π C_1. It is an arch lying in the first quadrant running in clockwise direction. Together with the line segment C_2 from $(2\pi, 0)$ to $(0, 0)$, it forms a simple closed curve running in clockwise direction. We have

$$\int_{C_1} xdy - ydx = \int_0^{2\pi} [(t - \sin t)(\sin t) - (1 - \cos t)(\cos t)]\, dt = \cdots = -6\pi \ .$$

On the other hand, $-C_2$ is parametrized by $c_2(x) = x\mathbf{i} + 0\mathbf{j}, x \in [0, 2\pi]$. We have $ydx - xdy = 0\, dx - x0\, dx = 0$ along it, hence

$$\int_{C_2} xdy - ydx = 0 \ .$$

According to the area formula

$$\text{Area} = \frac{1}{2}\left(\int_{-C_1} + \int_{-C_2}\right)(xdy - ydx)$$

the area is given by 3π.

no 35. Solution. Take $M = 0$ and $N = x^2/2$ in Green's theorem to get

$$\oint_C x^2/2dy = \iint_R xdA,$$

hence

$$\bar{x} \equiv \frac{1}{A}\iint_R xdA = \frac{1}{2A}\oint_C xdy \ .$$

no 37. Solution. By Green's theorem,

$$\oint_C (f_y dx - f_x dy) = \iint_R (-f_{xx} - f_{yy})\, dA = 0,$$

since f satisfies the Laplace's equation.

Supplementary Problems

1. Let D be the parallelogram formed by the lines $x+y = 1, x+y = 3, y = 2x-3, y = 2x+2$. Evaluate the line integral

$$\oint_C dx + 3xy\, dy$$

where C is the boundary of D oriented in anticlockwise direction. Suggestion: Try Green's theorem and then apply change of variables formula.

Solution. By Green's theorem

$$\oint_C dx + 3xy\, dy = \iint_D 3y\, dA(x, y) \ .$$

Next we apply the change of variables formula to evaluate this integral. Let $u = x + y$ and $v = y - 2x$. Then $(u, v) \mapsto (x, y)$ sends the rectangle $R = [1, 3] \times [-3, 2]$ to D. We have

$\dfrac{\partial(u,v)}{\partial(x,y)} = 3$ and $x = (u-v)/3$ and $y = (2u+v)/3$. By the change of variables formula

$$\begin{aligned}\iint_D 3y dA(x,y) &= \iint_R (2u+v)\frac{1}{3}\, dA(u,v) \\ &= \frac{1}{3}\int_1^3 \int_{-3}^2 (2u+v)\, dvdu \\ &= \frac{1}{3}\int_1^3 (10u-5)\, du \\ &= \frac{35}{3}\ .\end{aligned}$$

2. Find a potential for the vector field

$$\frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}\ ,$$

in the region obtained by deleting the line $(x,0), x \le 0$, from $\mathbb{R}^2$.

Solution. From $\dfrac{\partial \Phi}{\partial y} = \dfrac{x}{x^2+y^2}$, etc we get

$$\Phi(x,y) = \tan^{-1}\frac{y}{x}\ .$$

This is the argument, that is, the angle between (x,y) and the positive x-axis.

If you start with $\dfrac{\partial \Phi}{\partial x} = \dfrac{-y}{x^2+y^2}$, you get

$$\Phi(x,y) = -\tan^{-1}\frac{x}{y}\ ,$$

which is the same as the first one after observing the relation $\tan(\pi/2-\theta) = -1/\tan\theta$.

3. Let $F = M\mathbf{i} + N\mathbf{j}$ be a smooth vector field in $\mathbb{R}^2$ except at the origin. Suppose that $M_y = N_x$. Show that for any simple closed curve γ enclosing the origin and oriented in anticlockwise direction, one has

$$\oint_\gamma Mdx + Ndy = \varepsilon\int_0^{2\pi} [-M(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta,\varepsilon\sin\theta)\cos\theta]\, d\theta\ ,$$

for all sufficiently small ε. What happens when γ does not enclose the origin?

Solution. Let γ_ε be the circle entered at the origin with radius ε which is so small to be enclosed by γ. Then the vector field $\mathbf{F}$ is smooth in the region bounded by γ and γ_1. Applying Green's theorem in a multi-connected region we have

$$\oint_\gamma Mdx + Ndy = \oint_{\gamma'} Mdx + Ndy\ .$$

Using the standard parametrization, $\theta \mapsto (\varepsilon\cos\theta, \varepsilon\sin\theta)$, we further have

$$\oint_{\gamma'} Mdx + Ndy\ = \varepsilon\int_0^{2\pi} [-M(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta,\varepsilon\sin\theta)\cos\theta]\, d\theta\ ,$$

for all sufficiently small ε.

When the closed curve does not enclose the origin, the vector field is well-defined inside the curve. Applying the Green's theorem to the region bounded by this curve and use the condition $N_x - M_Y = 0$, we see that line integral vanishes.