Solution 8

Supplementary Problems

1. Let $F = (F_1, \cdots, F_n)$ be a smooth vector field in an open region in $\mathbb{R}^n$. Show that if it is conservative, then the necessary conditions (Component Test) hold

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \ , \quad \forall i, j.$$

Solution. Let $F = \nabla\Phi$. Then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial \Phi}{\partial x_i},$$

and

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial \Phi}{\partial x_j},$$

so they are equal. When $n = 3$, this reduces to the usual compatibility conditions (or necessary conditions, or component test):

$$M_z = P_x, \quad M_y = N_x, \quad N_z = P_y \ .$$

2. Let $\mathbf{F}$ be a smooth vector field in the entire space $\mathbb{R}^n$. Show that

$$\Phi(x, y, z) = \int_0^1 \mathbf{F}(tx, ty, tz) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\, dt \ ,$$

defines a potential function for $\mathbf{F}$ provided it passes the component test.

Solution. In a general dimension, the component test becomes

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \ ,$$

for different $i, j = 1, 2, \cdots, n$. With the above formula for Φ,

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_i} &= \int_0^1 [\frac{\partial F_1}{\partial x_i}(t\mathbf{x})tx_1 + \frac{\partial F_2}{\partial x_i}(t\mathbf{x})tx_2 + \cdots + \frac{\partial F_n}{\partial x_i}(t\mathbf{x})tx_n + F_i(t\mathbf{x})]\, dt \\
&= \int_0^1 [\frac{\partial F_i}{\partial x_1}(t\mathbf{x})tx_1 + \frac{\partial F_i}{\partial x_2}(t\mathbf{x})tx_2 + \cdots + \frac{\partial F_i}{\partial x_n}(t\mathbf{x})tx_n + F_i(t\mathbf{x})]\, dt \\
&= \int_0^1 \frac{d}{dt} tF_i(t\mathbf{x})\, dt \\
&= tF_i(t\mathbf{x})\Big|_0^1 \\
&= F_i(\mathbf{x}) \ .
\end{aligned}$$

Note. We presented in class a method of finding the potential function by successive integration. This formula provides another method, but I like the old approach better since you do not have to remember anything.

3. Let C be the oriented curve runs from the origin to $(2, 0)$ along the cardioid $r = 1 + \cos\theta$ in the upper half plane. Find the work done of $\mathbf{F} = (\sin xy + xy\cos xy)\mathbf{i} + x^2\cos xy\mathbf{j}$ along C.

   Solution 1. The vector field is conservative. In fact, by direct integration its potential is given by $\Phi(x, y) = x\sin xy$. Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Phi((2,0)) - \Phi((0,0)) = 0 \ .$$

   Solution 2. We verify

$$M_y = 2x\cos sy - x^2 y \sin xy = N_x \ .$$

   As the vector field is defined in the whole plane, $\mathbf{F}$ is conservative. (This follows from the previous problem or from Green's theorem, see next lecture.) The line integral from the origin to $(2, 0)$ along the cardioid is equal to the origin to $(2, 0)$ along the horizontal line segment $\mathbf{c}(t) = t\mathbf{i}, t \in [0, 2]$. Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (M(t,0)\mathbf{i} + N(t,0)\mathbf{j}) \cdot (t\mathbf{i})\, dt = 0 \ .$$

   Note. Here we take advantage of the conservative property of the vector field to avoid integration over the cardioid. In the second approach we avoid finding the potential, instead working on a simpler path.