# Solution 7

### Supplementary Problems

1. Let $f$ be a function on $[a,b]$. Verify that the parametric curve $x \mapsto x\mathbf{i} + f(x)\mathbf{j}$ is regular provided $f$ is continuously differentiable on $(a,b)$.

   **Solution.** Let the curve be $\mathbf{c}(x) = x\mathbf{i} + f(x)\mathbf{j}$. We have $\mathbf{c}'(t) = \mathbf{i} + f'(x)\mathbf{j}$ and

   $$|\mathbf{c}'(t)| = \sqrt{1 + (f'(x))^2} > 0 \ ,$$

   hence $\mathbf{c}$ is regular.

2. Let $\mathbf{c}$ be a regular parametric curve on $[a,b]$. Find a parametric curve $\gamma$ whose image is the same as $\mathbf{c}$ but reverse the orientation.

   **Solution.** Define

   $$\gamma(t) = \mathbf{c}(a+b-t) \qquad t \in [a,b] \ .$$

   Then $\gamma(a) = \mathbf{c}(b)$ and $\gamma(b) = \mathbf{c}(a)$. Moreover, $\gamma'(t) = -\mathbf{c}'(a+b-t)$ so $|\gamma'(t)| = |\mathbf{c}'(a+b-t)| > 0$, $\gamma$ is a regular parametric curve.

3. Let $\mathbf{c}$ be a parametric curve from $[a,b]$ to $C$. Another parametric curve $\gamma$ is called a reparametrization of $\mathbf{c}$ if $\gamma(t) = \mathbf{c}(\varphi(t))$ where $\varphi$ is a continuously differentiable map from $[\alpha,\beta]$ one-to-one onto $[a,b]$. Show that

   $$\int_a^b f(\mathbf{c}(t))|\mathbf{c}'(t)|\,dt = \int_\alpha^\beta f(\gamma(t))|\gamma'(t)|\,dt \ .$$

   **Solution.** From the relation $\gamma(t) = \mathbf{c}(\varphi(t))$ we have

   $$\gamma'(t) = \mathbf{c}'(\varphi(t))\varphi'(t) \ .$$

   First consider the case $\varphi$ maps $\alpha$ to $a$ and $\beta$ to $b$. Then $\varphi' > 0$. We have

   $$\begin{aligned}\int_\alpha^\beta f(\gamma(t))|\gamma'(t)|\,dt &= \int_\alpha^\beta f(\mathbf{c}(\varphi(t))|\mathbf{c}'(\varphi(t))\varphi'(t)|\,dt \\ &= \int_a^b f(\mathbf{c}(\tau))|\mathbf{c}'(\tau)|\,d\tau \quad (\text{letting } \tau = \varphi(t))\end{aligned}$$

   When $\varphi$ maps $\alpha$ to $b$ and $\beta$ to $a$, $\varphi' < 0$. We have

   $$\begin{aligned}\int_\alpha^\beta f(\gamma(t))|\gamma'(t)|\,dt &= \int_\alpha^\beta f(\mathbf{c}(\varphi(t))|\mathbf{c}'(\varphi(t))\varphi'(t)|\,dt \\ &= \int_b^a f(\mathbf{c}(\tau))|\mathbf{c}'(\tau)|(-1)\,d\tau \quad (\text{letting } \tau = \varphi(t)) \\ &= \int_a^b f(\mathbf{c}(\tau))|\mathbf{c}'(\tau)|\,d\tau \ .\end{aligned}$$

   **Note.** It was explained in class that the line integral of functions is independent of parametrization based on the Riemann sum approach. Here a more rigorous direct proof is present.