Solution 10

Supplementary Problems

1. Consider the parametric surface

$$\mathbf{r}(u,v) = (u+6v, -2u-12v+5, -1), \quad (u,v) \in [0,1] \times [0,1] \ .$$

Is it a smooth surface? Describe its image. Recall that by definition a parametric surface is smooth if $\mathbf{r}$ is continuously differentiable and $\mathbf{r}_u \times \mathbf{r}_v$ is linearly independent in the interior of the region of definition.

Solution. $\mathbf{r}_u \times \mathbf{r}_v = (1,-2,0) \times (6,-12,0) = 0$, hence this parametric surface is not smooth (or regular). In fact, the image of this parametric surface is $(u+6v, -2u-12v+5, -1) = (0,5,-1) + (u+6v)(1,-2,0)$. As $u+6v$ runs through all real numbers, the image is just the straight line $(0,5,-1) + t(1,-2,0)$, $t \in \mathbb{R}$.

Note. This example shows how the regular condition $|\mathbf{r}_u \times \mathbf{r}_v| > 0$ works.

2. Let S be the graph $\{(x,y,f(x,y)) : \ (x,y) \in D\}$ where D is a plane region. Show that its surface area is given by

$$\iint_D \sqrt{1+f_x^2+f_y^2}\, dA(x,y) \ .$$

Solution. S is parametrized by $\mathbf{r}(x,y) = (x,y,f(x,y))$. By a direct computation (done in class)

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1+f_x^2+f_y^2} \ .$$

Hence its surface area is given by

$$\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \iint_D \sqrt{1+f_x^2+f_y^2}\, dA(x,y) \ .$$

3. Let S be the surface of revolution obtained by rotating $(\varphi(z), z), \varphi(z) > 0, z \in [a,b]$ around the z-axis. Show that its surface area is given by

$$2\pi \int_a^b \varphi(z) \sqrt{1+\varphi'^2(z)}\, dz \ .$$

Solution. The parametrization of S is given by $\mathbf{r}(\theta, z) = (\varphi(z)\cos\theta, \varphi(z)\sin\theta, z), \theta \in [0,2\pi], z \in [a,b]$. By a direct computation (done in class) $|\mathbf{r}_u \times \mathbf{r}_v| = \varphi(z)\sqrt{1+\varphi'^2(z)}$. Hence, the surface area of S is given by

$$\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_0^{2\pi} \int_a^b \varphi(z)\sqrt{1+\varphi'(z)}\, dz d\theta = 2\pi \int_a^b \varphi(z)\sqrt{1+\varphi'^2(z)}\, dz \ .$$