Selected Solution to Assignment 1

No 34. Use Fubini's theorem to evaluate $\int_0^1\int_0^3 xe^{xy}\,dxdy$.

Solution. The evaluation is simpler by reversing the order of integration. In fact,

$$\begin{aligned}\int_0^1\int_0^3 xe^{xy}\,dxdy &= \int_0^3\int_0^1 xe^{xy}\,dydx \\ &= \int_0^3\int_0^1 \frac{d}{dy}e^{xy}\,dydx \\ &= \int_0^3 e^{xy}\Big|_0^1\,dx \\ &= \int_0^3 (e^x-1)\,dx \\ &= e^3-4\ .\end{aligned}$$

Supplementary Problems

1. Show that the function $\varphi(x)=1/x, x\in(0,1]$, and $\varphi(0)=1$ is not integrable on $[0,1]$.

   Solution Suppose on the contrary that φ is integrable. For all partitions with small norm $\|P\|$, their associated Riemann sums should come close to the same number

   $$I=\int_0^1 \varphi(x)\,dx,$$

   regardless of the tags chosen. However, consider an arbitrary partition with tags $\{z_j\}$. The Riemann sum

   $$\begin{aligned}S(\varphi,P) &= \sum_{j=1}^n \varphi(z_j)\Delta x_j \\ &= \frac{1}{z_1}\Delta x_1+\sum_{j=2}^n\frac{1}{z_j}\Delta x_j \\ &\geq \frac{1}{z_1}\Delta x_1\ .\end{aligned}$$

   While $z_j, j\geq 2$ are fixed, if we let z_1 becomes very small, $1/z_1$ becomes very large, so $S(\varphi,P)$ could become arbitrarily large and cannot come close to I. Therefore, φ cannot be integrable.

   Note In fact, it can be shown that all unbounded functions are non-integrable.

2. Suppose f is a non-negative function satisfying $\iint_R f(x,y)\,dA=0$. Does it imply that f is zero everywhere?

   Solution. When f is a non-negative integrable function, $\iint_R f=0$ does not necessarily imply f is equal to 0 everywhere. For instance, a function which vanishes everywhere except at finitely many points has zero integral. But, it is not the zero function.

   Note On the other hand, a non-negative *continuous* function which has zero integral must be the zero function.