Tutorial 3 2022/10/12

3.1 Average

When we calculate the average of the scores, we sum the scores of every students and calculate the quotient of the sum by the number of students. If we see that the integration is a kind of "sum" in a generalized sense, then the average we met in this course is nothing different as the the original ones that we met in high schools.

It can also have another explanation. Let $D \subset \mathbb{R}^2$ be a domain and f be an integrable function over D. Let's choose n points in D randomly, say they are $x_1, x_2, \cdots, x_n$. Then we calculate the average value $\overline{f}$ of f over $x_1, x_2, \cdots, x_n$ as follows

$$\overline{f} = \frac{\sum_{i=1}^{n} f(x_i)}{n}$$

When n is large enough, we would hope the average will approach the average of f over D, $\frac{1}{|D|}\iint_D f dA$. That is the following equality holds

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(x_i)}{n} = \frac{1}{|D|}\iint_D f dA.$$

There is still something that is not clear enough, what do we mean by choosing points in D "randomly"? It can mean any points have the equal probability to be chosen from D. To be more precise, let R be any subdomain of D, then we require the infinite sequence $\{x_i\}_{i\in\mathbb{N}}$ we have chosen should satisfy

$$\lim_{n\to\infty} \frac{|\{x_1, \ldots, x_n\} \cap R|}{n} = \frac{|R|}{|D|}.$$

We say the sequence $\{x_i\}_{i\in\mathbb{N}}$ that satisfies the above equation for any $R \subset D$ is **equidistributed** on D.

In fact we have

Theorem 3.1

A sequence of points $(x_1, x_2, \cdots)$ in D is equidistributed if and only if for every integrable function f we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(x_i)}{n} = \frac{1}{|D|}\iint_D f dA.$$

♡

Besides discrete points, we could also consider the curves running in the domain D "randomly" or "equidistributedly". Let $c : \mathbb{R}_{\geq 0} \to D \subset \mathbb{R}^2$ be such a curve. Then we can also make the following equality make sense

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(c(t))dt = \frac{1}{|D|}\iint_D f dA.$$

In the so called Ergodic theory, we call the left hand side of the equality the **time average** of f and the right hand side the **space average** of f.

3.2 Symmetry

The above discussion shows a way to calculate the integral of f over a domain D. Which would also be pretty consistent with our intuition.

Let's look at an example, the supplementary problems 3 in the exercise sheet 4:

Problem 3.1 Let D be a region in the plane which is symmetric with respect to the origin, that is, $(x,y) \in D$ if and only if $(-x,-y) \in D$. Show that

$$\iint_D f(x,y)dA(x,y) = 0,$$

when f is odd, that is, $f(-x,-y) = -f(x,y)$ in D. Suggestion: Use polar coordinates.

Proof We give two proof, one based on the hint and one based on theorem 3.1.

1. Solution. Let $\tilde{f}$ be the universal extension of f. It is readily checked that $\tilde{f}$ is an odd function in the entire plane. Let D_1 be a large disk of radius R centered at the origin containing D. By converting to polar coordinates,

$$\begin{aligned}\iint_D f &= \iint_{D_1} \tilde{f} dA \\ &= \int_0^{2\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta \\ &= \int_0^{\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta + \int_{\pi}^{2\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta.\end{aligned}$$

Further, using the change of variables $\alpha = \theta - \pi$, the second integral becomes

$$\begin{aligned}\int_{\pi}^{2\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta &= \int_0^{\pi}\int_0^R \tilde{f}(r\cos(\alpha+\pi), r\sin(\alpha+\pi)) r dr d\alpha \\ &= \int_0^{\pi}\int_0^R \tilde{f}(-r\cos\alpha, -r\sin\alpha) r dr d\alpha \\ &= -\int_0^{\pi}\int_0^R \tilde{f}(r\cos\alpha, r\sin\alpha) r dr d\alpha\end{aligned}$$

It follows that

$$\begin{aligned}\iint_D f &= \iint_{D_1} \tilde{f} dA \\ &= \int_0^{\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta + \int_{\pi}^{2\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta \\ &= \int_0^{\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta - \int_0^{\pi}\int_0^R \tilde{f}(r\cos\alpha, r\sin\alpha) r dr d\alpha = 0.\end{aligned}$$

2. Choose a equidistributed sequence $(p_1, p_2, \cdots)$ of points in D, where $p_i = (x_i, y_i) \in D \subset \mathbb{R}^2$. Let $q_i = -p_i = (-x_i, -y_i)$, then the sequence of points $(p_1, q_1, p_2, q_2, \cdots)$ is still equidistributed. By theorem 3.1, we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n f(p_i) + \sum_{i=1}^n f(q_i)}{2n} = \frac{1}{|D|}\iint_D f dA.$$

We have $f(p_i) + f(q_i) = f(p_i) + f(-p_i) = f(p_i) - f(p_i) = 0$, so the right hand side equal to 0. (You may not use this method in the exams unless you write down the proof of theorem 3.1. You may find the proof by searching the term "equidistributed sequence".)

You can prove similar results in the similar manner.

Problem 3.2 Let D be a region in the plane which is symmetric with respect to the x-axis, that is, $(x,y) \in D$ if and only if $(x,-y) \in D$. Show that

$$\iint_D f(x,y)dA(x,y) = 0,$$

when f is odd in the y-direction, that is, $f(x,-y) = -f(x,y)$ in D.

Problem 3.3 Let T be map on $\mathbb{R}^2$ which will rotate $\mathbb{R}^2$ by $\frac{2\pi}{3}$. So $T(x,y) = \left(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)$.

Let D be a region in the plane that $(x,y) \in D$ if and only if $T(x,y) \in D$.

Let f_1 and f_2 be two functions such that

$$f_1(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y) = -\frac{1}{2}f_1(x,y) - \frac{\sqrt{3}}{2}f_2(x,y)$$

$$f_2(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y) = \frac{\sqrt{3}}{2}f_1(x,y) - \frac{1}{2}f_2(x,y)$$

Show that

$$\iint_D f_1(x,y)dA(x,y) = \iint_D f_2(x,y)dA(x,y) = 0,$$

3.3 Mean Motion[1]

Conversely, theorem 3.1 can also be used to calculate some kind of limits which are hard to calculate through routine methods, by doing multiple integration.

Let us now turn our attention to the universe. Let's assume the sun, the earth and the moon are lying in the same plane. After we know the angular velocity of the earth rotating around sun and the angular velocity of the moon rotating around the earth, it is natural to ask what is the average angular velocity of the moon rotating around the sun.

The mathematics model for this problem is as follows. Let a_1, a_2 be two positive numbers represent the distance between the sun and the earth and the moon and the earth. Let ω_1 and ω_2 be the corresponding angle velocities. Then the average angular velocity of the moon around the sun is

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t})$$

Here i denotes the complex number $\sqrt{-1}$ and Arg denotes the variation of the total angle of the particle $a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$ since time $t = 0$.

When $a_1 = a_2 = a$, we can calculate the limit directly.

$$\begin{aligned}
&a e^{i\omega_1 t} + a e^{i\omega_2 t} \\
&= a\left(\cos\omega_1 t + \cos\omega_2 t\right) + i\left(\sin\omega_1 t + \sin\omega_2 t\right) \\
&= 2a\left(\cos\frac{\omega_1+\omega_2}{2}t\cos\frac{w_1-w_2}{2}t + i\sin\frac{w_1+w_2}{2}t\cos\frac{w_1-w_2}{2}t\right) \\
&= 2a\cos^{\frac{w_1-w_2}{2}t} e^{\frac{w_1+w_2}{2}t}
\end{aligned}$$

So

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(e^{\frac{w_1+w_2}{2}t}) = \frac{w_1+w_2}{2}$$

The average angular velocity is the average of ω_1 and ω_2.

For $a_1 \neq a_2$, things will become much more difficult. A brilliant method Weyl used in his paper *Mean Motion* is to turn it into the calculation of space average.

[1]The reference for this section is Herman Weyl's paper *Mean Motion*, published in 1938, American Journal of Mathematics

Let f be a function on $\mathbb{R}^2$ defined as follows

$$f(\theta_1, \theta_2) = \left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)}\right)$$

Since a modification of the starting angle will not affect the average velocity, we have

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)})$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(\theta_1 + \omega_1 \tau, \theta_2 + \omega_2 \tau) d\tau$$

According to the definition f is period such that $f(x,y) = f(x+2\pi, y) = f(x, y+2\pi)$, so we may view $(\omega_1 \tau, \omega_2 \tau)$ as a curve in $[0, 2\pi]^2$. As the time average equals the space average, we have

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(\theta_1, \theta_2)\, d\theta_1 d\theta_2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\theta_1 + \omega_1 t, \theta_2 + \omega_2 t)\, dt$$

Combine all of the above and we turned the calculation of limit into the calculation of the integral

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)}\right) d\theta_1 d\theta_2$$

Since

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)}\right) = \left.\frac{d}{dt}\right|_{t=0} \tan^{-1}\left(\frac{a_1 \sin(\theta_1 + w_1 t) + a_2 \sin(\theta_2 + w_2 t)}{a_1 \cos(\theta_1 + w_1 t) + a_2 \cos(\theta_2 + w_2 t)}\right)$$

by estimatedly calculation we can assume

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)}\right) = p_1(\theta_1, \theta_2)\omega_1 + p_2(\theta_1, \theta_2)\omega_2$$

Let

$$\overline{p_i} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} p_i(\theta_1, \theta_2) d\theta_1 d\theta_2$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \omega_1 \overline{p_1} + \omega_2 \overline{p_2}$$

To calculate $\overline{p_1}$, let $\omega_1 = 1, \omega_2 = 0$, then

$$\begin{aligned}
\overline{p_1} &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + t)} + a_2 e^{i\theta_2}\right) d\theta_1 d\theta_2 \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{d}{d\theta_1} \operatorname{Arg}\left(a_1 e^{i\theta_1} + a_2 e^{i\theta_2}\right) d\theta_1 d\theta_2
\end{aligned}$$

And

$$\begin{aligned}
&\int_0^{2\pi} \frac{d}{d\theta_1} \operatorname{Arg}\left(a_1 e^{i\theta_1} + a_2 e^{i\theta_2}\right) d\theta_1 \\
&= \operatorname{Arg}\left(a_1 e^{i\theta_1} + a_2 e^{i\theta_2}\right)\Big|_{a_1=0}^{\theta_1=2\pi} \\
&= \begin{cases} 2\pi, & a_1 > a_2 \\ 0, & a_1 < a_2 \end{cases}
\end{aligned}$$

Therefore we have $\overline{p_1} = 1, \overline{p_2} = 0$ if $a_1 > a_2$ or $\overline{p_2} = 1, \overline{p_1} = 0$ if $a_2 > a_1$.

To conclude, we see that the average angular velocity only depend on angular velocity of the particle with longer radius. So we proved that the average angular velocity of the moon rotating about the sun is the same as the angular velocity of the earth.

Remark The above argument requires that $\frac{\omega_1}{\omega_2}$ is not a rational number in order to guarentee that the path $(\omega_1 t, \omega_2 t) \mod 2\pi$ is equidistributed on $[0, 2\pi]^2$. If $\frac{\omega_1}{\omega_2}$ is rational, the result still holds as the limit should be continuous on (ω_1, ω_2).

✍ **Exercise 3.1** Calculate

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t})$$

Show that if a_1, a_2, a_3 form a triangle whose corresponding interior angles are $\alpha_1, \alpha_2, \alpha_3$, then

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t}) = \frac{\alpha_1}{\pi}\omega_1 + \frac{\alpha_2}{\pi}\omega_2 + \frac{\alpha_3}{\pi}\omega_3$$

✍ **Exercise 3.2** Say something about

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t} + a_4 e^{i\omega_4 t})$$

✍ **Exercise 3.3** Prove that the mean of the square of the length $\lim_{t\to\infty} \frac{1}{T}\int_0^T |\sum_{k=1}^N a_k e^{i\omega_k t}|^2 dt$ is $\sum_{k=1}^N |a_k|^2$. (Hint: By theorem 3.1 we have $\lim_{t\to\infty} \frac{1}{T}\int_0^T |\sum_{k=1}^N a_k e^{i\omega_k t}|^2 dt = \frac{1}{(2\pi)^N}\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} |\sum_{k=1}^N a_k e^{i\theta_k}|^2 d\theta_1 d\theta_2 \cdots d\theta_N$ and $|\sum_{k=1}^N a_k e^{i\theta_k}|^2 = \sum_{k=1}^N |a_k|^2 + \sum_{j=1}^N \sum_{l=1}^N a_j a_l e^{i(\theta_j - \theta_l)}$)