Exercises 15.2

Sketching Regions of Integration

In Exercises 1–8, sketch the described regions of integration.

1. $0 \le x \le 3, \quad 0 \le y \le 2x$

2. $-1 \le x \le 2, \quad x - 1 \le y \le x^2$

3. $-2 \le y \le 2, \quad y^2 \le x \le 4$

4. $0 \le y \le 1, \quad y \le x \le 2y$

5. $0 \le x \le 1, \quad e^x \le y \le e$

6. $1 \le x \le e^2, \quad 0 \le y \le \ln x$

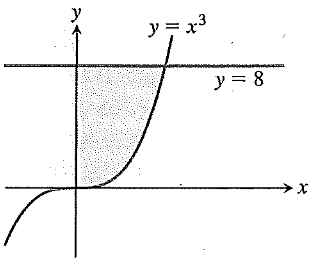
7. $0 \le y \le 1, \quad 0 \le x \le \sin^{-1} y$

8. $0 \le y \le 8, \quad \frac{1}{4}y \le x \le y^{1/3}$

Finding Limits of Integration

In Exercises 9–18, write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

9.



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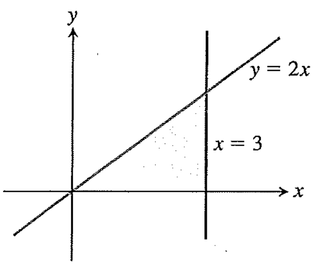
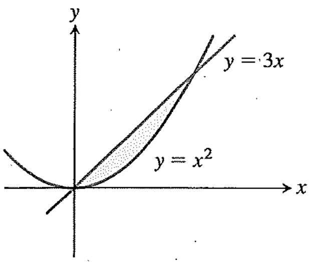
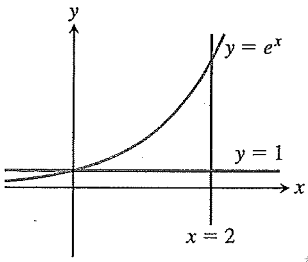


11.



12.



13. Bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$

14. Bounded by $y = \tan x$, $x = 0$, and $y = 1$

15. Bounded by $y = e^{-x}$, $y = 1$, and $x = \ln 3$

16. Bounded by $y = 0$, $x = 0$, $y = 1$, and $y = \ln x$

17. Bounded by $y = 3 - 2x$, $y = x$, and $x = 0$

18. Bounded by $y = x^2$ and $y = x + 2$

Finding Regions of Integration and Double Integrals

In Exercises 19–24, sketch the region of integration and evaluate the integral.

19. $\displaystyle\int_0^{\pi}\int_0^{x} x \sin y \, dy \, dx$

20. $\displaystyle\int_0^{\pi}\int_0^{\sin x} y \, dy \, dx$

21. $\displaystyle\int_1^{\ln 8}\int_0^{\ln y} e^{x+y} \, dx \, dy$

22. $\displaystyle\int_1^{2}\int_y^{y^2} dx \, dy$

23. $\displaystyle\int_0^{1}\int_0^{y^2} 3y^3 e^{xy} \, dx \, dy$

24. $\displaystyle\int_1^{4}\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} \, dy \, dx$

In Exercises 25–28, integrate f over the given region.

25. Quadrilateral $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, and $x = 2$

26. Triangle $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$

27. Triangle $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$

28. Curved region $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$

Each of Exercises 29–32 gives an integral over a region in a Cartesian coordinate plane. Sketch the region and evaluate the integral.

29. $\displaystyle\int_{-2}^{0}\int_v^{-v} 2 \, dp \, dv$ (the pv-plane)

30. $\displaystyle\int_0^{1}\int_0^{\sqrt{1-s^2}} 8t \, dt \, ds$ (the st-plane)

31. $\displaystyle\int_{-\pi/3}^{\pi/3}\int_0^{\sec t} 3 \cos t \, du \, dt$ (the tu-plane)

32. $\displaystyle\int_0^{3/2}\int_1^{4-2u} \frac{4 - 2u}{v^2} \, dv \, du$ (the uv-plane)

Reversing the Order of Integration

In Exercises 33–46, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

33. $\displaystyle\int_0^{1}\int_2^{4-2x} dy \, dx$

34. $\displaystyle\int_0^{2}\int_{y-2}^{0} dx \, dy$

35. $\displaystyle\int_0^{1}\int_y^{\sqrt{y}} dx \, dy$

36. $\displaystyle\int_0^{1}\int_{1-x}^{1-x^2} dy \, dx$

37. $\displaystyle\int_0^{1}\int_1^{e^x} dy \, dx$

38. $\displaystyle\int_0^{\ln 2}\int_{e^y}^{2} dx \, dy$

39. $\displaystyle\int_0^{3/2}\int_0^{9-4x^2} 16x \, dy \, dx$

40. $\displaystyle\int_0^{2}\int_0^{4-y^2} y \, dx \, dy$

41. $\displaystyle\int_0^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y \, dx \, dy$

42. $\displaystyle\int_0^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x \, dy \, dx$

43. $\displaystyle\int_1^{e}\int_0^{\ln x} xy \, dy \, dx$

44. $\displaystyle\int_0^{\pi/6}\int_{\sin x}^{1/2} xy^2 \, dy \, dx$

45. $\displaystyle\int_0^{3}\int_1^{e^y} (x + y) \, dx \, dy$

46. $\displaystyle\int_0^{\sqrt{3}}\int_0^{\tan^{-1} y} \sqrt{xy} \, dx \, dy$

In Exercises 47–56, sketch the region of integration, reverse the order of integration, and evaluate the integral.

47. $\displaystyle\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}\,dy\,dx$

48. $\displaystyle\int_0^{2}\int_x^{2}2y^2\sin xy\,dy\,dx$

49. $\displaystyle\int_0^{1}\int_y^{1}x^2e^{xy}\,dx\,dy$

50. $\displaystyle\int_0^{2}\int_0^{4-x^2}\frac{xe^{2y}}{4-y}\,dy\,dx$

51. $\displaystyle\int_0^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}}e^{x^2}\,dx\,dy$

52. $\displaystyle\int_0^{3}\int_{\sqrt{x/3}}^{1}e^{y^3}\,dy\,dx$

53. $\displaystyle\int_0^{1/16}\int_{y^{1/4}}^{1/2}\cos(16\pi x^5)\,dx\,dy$

54. $\displaystyle\int_0^{8}\int_{\sqrt[3]{x}}^{2}\frac{dy\,dx}{y^4+1}$

55. Square region $\iint_R (y-2x^2)\,dA$ where R is the region bounded by the square $|x|+|y|=1$

56. Triangular region $\iint_R xy\,dA$ where R is the region bounded by the lines $y=x$, $y=2x$, and $x+y=2$

Volume Beneath a Surface $z = f(x, y)$

57. Find the volume of the region bounded above by the paraboloid $z=x^2+y^2$ and below by the triangle enclosed by the lines $y=x$, $x=0$, and $x+y=2$ in the xy-plane.

58. Find the volume of the solid that is bounded above by the cylinder $z=x^2$ and below by the region enclosed by the parabola $y=2-x^2$ and the line $y=x$ in the xy-plane.

59. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y=4-x^2$ and the line $y=3x$, while the top of the solid is bounded by the plane $z=x+4$.

60. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2+y^2=4$, and the plane $z+y=3$.

61. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x=3$, and the parabolic cylinder $z=4-y^2$.

62. Find the volume of the solid cut from the first octant by the surface $z=4-x^2-y$.

63. Find the volume of the wedge cut from the first octant by the cylinder $z=12-3y^2$ and the plane $x+y=2$.

64. Find the volume of the solid cut from the square column $|x|+|y|\le 1$ by the planes $z=0$ and $3x+z=3$.

65. Find the volume of the solid that is bounded on the front and back by the planes $x=2$ and $x=1$, on the sides by the cylinders $y=\pm 1/x$, and above and below by the planes $z=x+1$ and $z=0$.

66. Find the volume of the solid bounded on the front and back by the planes $x=\pm\pi/3$, on the sides by the cylinders $y=\pm\sec x$, above by the cylinder $z=1+y^2$, and below by the xy-plane.

In Exercises 67 and 68, sketch the region of integration and the solid whose volume is given by the double integral.

67. $\displaystyle\int_0^{3}\int_0^{2-2x/3}\left(1-\frac{1}{3}x-\frac{1}{2}y\right)dy\,dx$

68. $\displaystyle\int_0^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}\sqrt{25-x^2-y^2}\,dx\,dy$

Integrals over Unbounded Regions

Improper double integrals can often be computed similarly to improper integrals of one variable. The first iteration of the following improper integrals is conducted just as if they were proper integrals. One then evaluates an improper integral of a single variable by taking appropriate limits, as in Section 8.8. Evaluate the improper integrals in Exercises 69–72 as iterated integrals.

69. $\displaystyle\int_1^{\infty}\int_{e^{-x}}^{1}\frac{1}{x^3y}\,dy\,dx$

70. $\displaystyle\int_{-1}^{1}\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}}(2y+1)\,dy\,dx$

71. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(y^2+1)}\,dx\,dy$

72. $\displaystyle\int_0^{\infty}\int_0^{\infty}xe^{-(x+2y)}\,dx\,dy$

Approximating Integrals with Finite Sums

In Exercises 73 and 74, approximate the double integral of $f(x, y)$ over the region R partitioned by the given vertical lines $x=a$ and horizontal lines $y=c$. In each subrectangle, use (x_k, y_k) as indicated for your approximation.

$$\iint_R f(x,y)\,dA \approx \sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k.$$

73. $f(x,y)=x+y$ over the region R bounded above by the semicircle $y=\sqrt{1-x^2}$ and below by the x-axis, using the partition $x=-1,-1/2,0,1/4,1/2,1$ and $y=0,1/2,1$ with (x_k,y_k) the lower left corner in the kth subrectangle (provided the subrectangle lies within R)

74. $f(x,y)=x+2y$ over the region R inside the circle $(x-2)^2+(y-3)^2=1$ using the partition $x=1,3/2,2,5/2,3$ and $y=2,5/2,3,7/2,4$ with (x_k,y_k) the center (centroid) in the kth subrectangle (provided the subrectangle lies within R)

Theory and Examples

75. Circular sector Integrate $f(x,y)=\sqrt{4-x^2}$ over the smaller sector cut from the disk $x^2+y^2\le 4$ by the rays $\theta=\pi/6$ and $\theta=\pi/2$.

76. Unbounded region Integrate $f(x,y)=1/[(x^2-x)(y-1)^{2/3}]$ over the infinite rectangle $2\le x<\infty$, $0\le y\le 2$.

77. Noncircular cylinder A solid right (noncircular) cylinder has its base R in the xy-plane and is bounded above by the paraboloid $z=x^2+y^2$. The cylinder's volume is

$$V=\int_0^1\int_0^y (x^2+y^2)\,dx\,dy+\int_1^2\int_0^{2-y}(x^2+y^2)\,dx\,dy.$$

Sketch the base region R and express the cylinder's volume as a single iterated integral with the order of integration reversed. Then evaluate the integral to find the volume.

78. Converting to a double integral Evaluate the integral

$$\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x)\, dx.$$

(*Hint:* Write the integrand as an integral.)

79. Maximizing a double integral What region R in the xy-plane maximizes the value of

$$\iint_R (4 - x^2 - 2y^2)\, dA?$$

Give reasons for your answer.

80. Minimizing a double integral What region R in the xy-plane minimizes the value of

$$\iint_R (x^2 + y^2 - 9)\, dA?$$

Give reasons for your answer.

81. Is it possible to evaluate the integral of a continuous function $f(x, y)$ over a rectangular region in the xy-plane and get different answers depending on the order of integration? Give reasons for your answer.

82. How would you evaluate the double integral of a continuous function $f(x, y)$ over the region R in the xy-plane enclosed by the triangle with vertices (0, 1), (2, 0), and (1, 2)? Give reasons for your answer.

83. Unbounded region Prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy = \lim_{b\to\infty}\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\, dx\, dy = 4\left(\int_0^{\infty} e^{-x^2}\, dx\right)^2.$$

84. Improper double integral Evaluate the improper integral

$$\int_0^1\int_0^3 \frac{x^2}{(y-1)^{2/3}}\, dy\, dx.$$

COMPUTER EXPLORATIONS

Use a CAS double-integral evaluator to estimate the values of the integrals in Exercises 85–88.

85. $\displaystyle\int_1^3\int_1^x \frac{1}{xy}\, dy\, dx$

86. $\displaystyle\int_0^1\int_0^1 e^{-(x^2+y^2)}\, dy\, dx$

87. $\displaystyle\int_0^1\int_0^1 \tan^{-1} xy\, dy\, dx$

88. $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2}\, dy\, dx$

Use a CAS double-integral evaluator to find the integrals in Exercises 89–94. Then reverse the order of integration and evaluate, again with a CAS.

89. $\displaystyle\int_0^1\int_{2y}^4 e^{x^2}\, dx\, dy$

90. $\displaystyle\int_0^3\int_{x^2}^9 x\cos(y^2)\, dy\, dx$

91. $\displaystyle\int_0^2\int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\, dx\, dy$

92. $\displaystyle\int_0^2\int_0^{4-y^2} e^{xy}\, dx\, dy$

93. $\displaystyle\int_1^2\int_0^{x^2} \frac{1}{x+y}\, dy\, dx$

94. $\displaystyle\int_1^2\int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy$

15.3 Area by Double Integration

In this section we show how to use double integrals to calculate the areas of bounded regions in the plane, and to find the average value of a function of two variables.

Areas of Bounded Regions in the Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \sum_{k=1}^{n} \Delta A_k. \tag{1}$$

This is simply the sum of the areas of the small rectangles in the partition of R, and approximates what we would like to call the area of R. As the norm of a partition of R approaches zero, the height and width of all rectangles in the partition approach zero, and the coverage of R becomes increasingly complete (Figure 15.8). We define the area of R to be the limit

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n} \Delta A_k = \iint_R dA. \tag{2}$$

Solution The value of the integral of f over R is

$$\int_0^{\pi}\int_0^1 x\cos xy\,dy\,dx = \int_0^{\pi}\left[\sin xy\right]_{y=0}^{y=1} dx \qquad \int x\cos xy\,dy = \sin xy + C$$

$$= \int_0^{\pi}(\sin x - 0)\,dx = -\cos x\Big]_0^{\pi} = 1 + 1 = 2.$$

The area of R is π. The average value of f over R is $2/\pi$. ■

ises 15.3

Double Integrals

cises 1–12, sketch the region bounded by the given lines and Then express the region's area as an iterated double integral luate the integral.

coordinate axes and the line $x + y = 2$

lines $x = 0, y = 2x$, and $y = 4$

parabola $x = -y^2$ and the line $y = x + 2$

parabola $x = y - y^2$ and the line $y = -x$

curve $y = e^x$ and the lines $y = 0, x = 0$, and $x = \ln 2$

curves $y = \ln x$ and $y = 2\ln x$ and the line $x = e$, in the t quadrant

parabolas $x = y^2$ and $x = 2y - y^2$

parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

lines $y = x, y = x/3$, and $y = 2$

lines $y = 1 - x$ and $y = 2$ and the curve $y = e^x$

lines $y = 2x, y = x/2$, and $y = 3 - x$

lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

ing the Region of Integration

grals and sums of integrals in Exercises 13–18 give the areas ns in the xy-plane. Sketch each region, label each bounding ith its equation, and give the coordinates of the points where es intersect. Then find the area of the region.

$\int \int_{y^2/3}^{2y} dx\,dy$ **14.** $\int_0^3\int_{-x}^{x(2-x)} dy\,dx$

$\int^{\pi/4}\int_{\sin x}^{\cos x} dy\,dx$ **16.** $\int_{-1}^2\int_{y^2}^{y+2} dx\,dy$

$\int \int_{-2x}^{1-x} dy\,dx + \int_0^2\int_{-x/2}^{1-x} dy\,dx$

$\int \int_{x^2-4}^{0} dy\,dx + \int_0^4\int_0^{\sqrt{x}} dy\,dx$

Average Values

d the average value of $f(x, y) = \sin(x + y)$ over

the rectangle $0 \le x \le \pi, \quad 0 \le y \le \pi$.

the rectangle $0 \le x \le \pi, \quad 0 \le y \le \pi/2$.

ich do you think will be larger, the average value of $, y) = xy$ over the square $0 \le x \le 1, 0 \le y \le 1$, or the average value of f over the quarter circle $x^2 + y^2 \le 1$ in the first quadrant? Calculate them to find out.

21. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, 0 \le y \le 2$.

22. Find the average value of $f(x, y) = 1/(xy)$ over the square $\ln 2 \le x \le 2\ln 2, \ln 2 \le y \le 2\ln 2$.

Theory and Examples

23. Geometric area Find the area of the region

$$R: 0 \le x \le 2,\ 2 - x \le y \le \sqrt{4 - x^2},$$

using **(a)** Fubini's Theorem, **(b)** simple geometry.

24. Geometric area Find the area of the circular washer with outer radius 2 and inner radius 1, using **(a)** Fubini's Theorem, **(b)** simple geometry.

25. Bacterium population If $f(x, y) = (10{,}000e^y)/(1 + |x|/2)$ represents the "population density" of a certain bacterium on the xy-plane, where x and y are measured in centimeters, find the total population of bacteria within the rectangle $-5 \le x \le 5$ and $-2 \le y \le 0$.

26. Regional population If $f(x, y) = 100\,(y + 1)$ represents the population density of a planar region on Earth, where x and y are measured in kilometers, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.

27. Average temperature in Texas According to the *Texas Almanac*, Texas has 254 counties and a National Weather Service station in each county. Assume that at time t_0, each of the 254 weather stations recorded the local temperature. Find a formula that would give a reasonable approximation of the average temperature in Texas at time t_0. Your answer should involve information that you would expect to be readily available in the *Texas Almanac*.

28. If $y = f(x)$ is a nonnegative continuous function over the closed interval $a \le x \le b$, show that the double integral definition of area for the closed plane region bounded by the graph of f, the vertical lines $x = a$ and $x = b$, and the x-axis agrees with the definition for area beneath the curve in Section 5.3.

29. Suppose $f(x, y)$ is continuous over a region R in the plane and that the area $A(R)$ of the region is defined. If there are constants m and M such that $m \le f(x, y) \le M$ for all $(x, y) \in R$, prove that

$$mA(R) \le \iint_R f(x, y)\,dA \le MA(R).$$

30. Suppose $f(x, y)$ is continuous and nonnegative over a region R in the plane with a defined area $A(R)$. If $\iint_R f(x, y)\,dA = 0$, prove that $f(x, y) = 0$ at every point $(x, y) \in R$.