Solution to Assignment 4

15.5

(24). The region is over a rectangle which can be decomposed into two trianlges D_1 and D_2. D_1 has vertices at $(0,0),(1,0),(1,2)$ and D_2 has vertices at $(0,0),(1,2),(0,2)$. Over D_1, the region is described by $0 \le z \le 1-x$. Over D_2, it is given by $0 \le z \le (2-y)/2$. Hence the volume of the region is

$$\iint_{D_1} \int_0^{1-x} 1\, dz\, dA(x,y) + \iint_{D_2} \int_0^{(2-y)/2} 1\, dz\, dA(x,y) \ = \cdots \ .$$

(27). Let $\mathbf{v}_1 = (0,2,0)-(1,0,0) = (-1,2,0)$ and $\mathbf{v}_2 = (0,0,3)-(1,0,0) = (-1,0,3)$. Then $(a,b,c) = \mathbf{v}_1 \times \mathbf{v}_2 = (6,3,2)$. The equation of the plane passing through $(1,0,0),(0,2,0),(0,0,3)$ is given by $6x+3y+2z = d$. Setting $(x,y,z) = (1,0,0)$, the equation is $6x+3y+2z = 6$. Regarding it as a region over the triangle T in the xy-plane with vertices at $(0,0),(1,0),(0,2)$, the volume of the tetrahedron is

$$\iint_T \int_0^{(6-6x-3y)/2} dz\, dA(x,y) = \cdots \ .$$

(29) The region is described by $0 \le z \le \sqrt{1-x^2}$ where (x,y) satisfies $x^2+y^2 \le 1, x,y \ge 0$. Therefore, the volume of this region is

$$8 \times \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} 1\, dz\, dydx = \cdots = 16/3 \ .$$

Supplementary Problems

1. Find the equations of the planes passing through the origin and (a) $(1,2,3),(0,-2,0)$ and (b) $(0,2,-1),(3,0,5)$.

   Solution. (a) $(1,2,3)\times(0,-2,0) = (6,0,-2)$. The equation is $6x-2z = 0$ or $3x - z = 0$.

   (b) $(0,2,-1)\times(3,0,5) = (10,-3,-6)$. The equation is $10x-3y-6z = 0$.

2. Find the equation of the plane passing the points $(1,0,-1),(4,0,0),(6,2,1)$.

   Soluton. Take $\mathbf{u}_0 = (4,0,0)$. (You can take any one of these three points as the base point.) Then $\mathbf{v}_1 = (1,0,-1)-(4,0,0) = (-3,0,-1)$, and $\mathbf{v}_2 = (6,2,1)-(4,0,0) = (2,2,1)$. $\mathbf{v}_1 \times \mathbf{v}_2 = (2,1,-6)$. The equation is $2x+y-6z = d$. Since $(4,0,0)$ belongs to the plane, $d = 2\times 4 + 0 - 6\times 0 = 8$. The equation of this plane is $2x+y-6z = 8$.

3. Let D be a region in the plane which is symmetric with respect to the origin, that is, $(x,y) \in D$ if and only if $(-x,-y) \in D$. Show that

$$\iint_D f(x,y)\, dA(x,y) = 0 \ ,$$

when f is odd, that is, $f(-x,-y) = -f(x,y)$ in D.

Solution. Let $\tilde{f}$ be the universal extension of f. It is readily checked that $\tilde{f}$ is an odd function in the entire plane. Let D_1 be a large disk of radius R centered at the origin containing D. By converting to polar coordinates,

$$\begin{aligned}\iint_D f &= \iint_{D_1} \tilde{f}\, dA \\ &= \int_0^{2\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta \\ &= \int_0^{\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta + \int_\pi^{2\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta\ .\end{aligned}$$

Further, using the change of variables $\alpha = \theta - \pi$, the second integral becomes

$$\begin{aligned}\int_\pi^{2\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta &= \int_0^{\pi}\int_0^R \tilde{f}(r\cos(\alpha+\pi), r\sin(\alpha+\pi)) r\, dr d\alpha \\ &= \int_0^{\pi}\int_0^R \tilde{f}(-r\cos\alpha, -r\sin\alpha) r\, dr d\alpha \\ &= -\int_0^{\pi}\int_0^R \tilde{f}(r\cos\alpha, r\sin\alpha) r\, dr d\alpha.\end{aligned}$$

It follows that

$$\begin{aligned}\iint_D f &= \iint_{D_1} \tilde{f}\, dA \\ &= \int_0^{\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta + \int_\pi^{2\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta \\ &= \int_0^{\pi}\int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta - \int_0^{\pi}\int_0^R \tilde{f}(r\cos\alpha, r\sin\alpha) r\, dr d\alpha = 0\ .\end{aligned}$$

Alternate Solution. Let $R = [-a,a]\times[-c,c]$ be a large rectangle which covers D. Let $D_1 = \{(x,y) :\ (x,y)\in D, y\ge 0\}$ and $D_2 = \{(x,y) :\ (x,y)\in D,\ y\le 0\}$. Then D_2 is contained in $[-a,a]\times[-c,0]$.

$$\begin{aligned}\iint_{D_2} f &= \int_{-a}^{a}\int_{-c}^{0} \tilde{f}(x,y)\, dydx \\ &= \int_{-a}^{a}\int_{c}^{0} \tilde{f}(x,-t)(-1) dtdx \quad (t=-y) \\ &= \int_{-a}^{a}\int_{0}^{c} \tilde{f}(x,-t)\, dtdx \\ &= -\int_{-a}^{a}\int_{0}^{c} \tilde{f}(-x,t)\, dtdx \\ &= -\int_{0}^{c}\int_{-a}^{a} \tilde{f}(-x,t)\, dxdt \\ &= -\int_{0}^{c}\int_{a}^{-a} \tilde{f}(s,t)(-1) dsdt \quad (s=-x) \\ &= -\int_{0}^{c}\int_{-a}^{a} \tilde{f}(s,t)\, dsdt \\ &= -\int_{-a}^{a}\int_{0}^{c} \tilde{f}(s,t)\, dtds\ .\end{aligned}$$

It follows that

$$\begin{aligned}\iint_D f = \iint_{D_1} f + \iint_{D_2} f &= \int_{-a}^{a}\int_{0}^{c} \tilde{f}(x,y)\,dydx + \int_{-a}^{a}\int_{c}^{0} \tilde{f}(x,y)\,dydx \\ &= \int_{-a}^{a}\int_{0}^{c} \tilde{f}(x,y)\,dydx - \int_{-a}^{a}\int_{0}^{c} \tilde{f}(s,t)\,dtds \\ &= 0\ .\end{aligned}$$