Solution to Assignment 3

29. Find the area of one leaf of the rose $r = 12\cos 3\theta$.

Solution. As the cosine function is 2π-periodic, $\cos 3\theta$ is $2\pi/3$-periodic. It suffices to plot its graph in $[-\pi/3, \pi/3]$. Observing that in this interval, $\cos 3\theta$ is non-negative only on $[-\pi/6, \pi/6]$, there is one leaf sitting in $[-\pi/6, \pi/6]$. By rotating it by $2\pi/3$ and then by $4\pi/3$, we obtain the full graph of the rose which consists of three identical leaves.

By symmetry, the area of one leaf is

$$\int_{-\pi/6}^{\pi/6}\int_0^{12\cos 3\theta} r\,drd\theta = 2\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,drd\theta = 12\pi \ .$$

41. In this problem we establish the famous formula by using double integral in a tricky way. Setting

$$a = \int_{-\infty}^{\infty} e^{-x^2}\,dx \ .$$

We have

$$\begin{aligned} a^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\ &= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA(x,y) \\ &= \lim_{R\to\infty}\iint_{D_R} e^{-x^2-y^2}\,dA(x,y) \\ &= \lim_{R\to\infty}\int_0^{2\pi}\int_0^{R} e^{-r^2} r\,drd\theta \\ &= \lim_{R\to\infty}\int_0^{2\pi}\int_0^{R^2} e^{-s}\,dsd\theta \\ &= \pi. \end{aligned}$$

Hence

$$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \ .$$

Supplementary Problems

1. Express the hyperbola $x^2 - y^2 = 1$ $(y \geq 0)$ in polar coordinates. What is the range of θ?

   Solution. From $1 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$ we get

   $$r = \frac{1}{\sqrt{\cos 2\theta}} \ ,$$

   where $\theta \in (-\pi/4, \pi/4)$.

2. Let D be the sector bounded by the line $y = ax, a > 0$, the positive y-axis and the circle $x^2 + y^2 = r^2$. Use cartesian coordinates in your integration to show that its area is given by $r^2\Theta/2$ where $\Theta = \tan^{-1} a \in (0, \pi/2)$.

Solution. Let $\tan\alpha = a$. α is uniquely determined in $(0, \pi/2)$. The point of intersection of the line and the circle is given by $x = r\cos\alpha$ and $y = r\sin\alpha$. D is described in cartesian coordinates by $ax \le y \le \sqrt{r^2 - x^2}, 0 \le x \le r\cos\alpha$. The area of the sector is

$$\begin{aligned}
\iint_D 1\,dA &= \int_0^{r\cos\alpha}\int_{ax}^{\sqrt{r^2-x^2}} dydx \\
&= \int_0^{r\cos\theta} (\sqrt{r^2-x^2} - ax)\,dx \\
&= \left(\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r} - a\frac{x^2}{2}\right)\Big|_0^{r\cos\alpha} \\
&= \frac{1}{2}r^2\cos\alpha\sin\alpha + \frac{1}{2}r^2\sin^{-1}\cos\alpha - \tan\alpha\frac{r^2\cos^2\alpha}{2} \\
&= \frac{1}{2}r^2\Theta\ ,
\end{aligned}$$

after using the relation $\cos\alpha = \sin\Theta$.

Note. This formula for the area of a sector has been used in the derivation of the basic formula

$$\iint_D f(x,y)\,dA(x,y) = \iint_R f(r\cos\theta, r\sin\theta) r dA(r,\theta)\ ,$$

where D is a sector and R is the corresponding rectangle. Although well-known since high school or even primary school, it is consistent to derive it here by double integral.

3. Let D be the region bounded by the graph of $y = \sqrt{1-x^2}+1$ and the x-axis over $0 \le x \le 1$. Describe it in polar coordinates.

Solution. As a polar curve, $x^2 + (y-1)^2 = 1$ is given by $r = 2\sin\theta$. D is the union of D_1 and D_2 where

$$D_1 : 0 \le \theta \le \pi/4,\ 0 \le r \le 1/\cos\theta\ ,$$

and

$$D_2 :\ \pi/4 \le \theta \le \pi/2,\ 1/\cos\theta \le r \le 2\sin\theta\ .$$