MATH 2010E Advanced Calculus I
Suggested Solution of Homework 3

Exercise 14.2

9. Find the limit $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$.

Solution.

$$\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = \lim_{(x,y)\to(0,0)} (e^y)\left(\frac{\sin x}{x}\right) = \lim_{(x,y)\to(0,0)} (e^y) \lim_{(x,y)\to(0,0)} \left(\frac{\sin x}{x}\right) = 1 \cdot 1 = 1.$$

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21. Find the limit $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ by rewriting the fraction first.

Solution. Let $r = \sqrt{x^2+y^2}$. Then

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1.$$

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27. Find the limit $\lim_{P\to(\pi,\pi,0)} \left(\sin^2 x + \cos^2 y + \sec^2 z\right)$.

Solution. $\lim_{P\to(\pi,\pi,0)} \left(\sin^2 x + \cos^2 y + \sec^2 z\right) = \sin^2 \pi + \cos^2 \pi + \sec^2 0 = 0+1+1 = 2.$ ◀

41. By considering different paths of approach, show that the function $f(x,y) = -\frac{x}{\sqrt{x^2+y^2}}$ has no limit as $(x,y) \to (0,0)$.

Solution. Along $x = 0$,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = \lim_{y\to 0} -\frac{0}{\sqrt{0^2+y^2}} = \lim_{y\to 0} 0 = 0.$$

Along $y = 0$ and $x > 0$,

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0,\ x>0}} f(x,y) = \lim_{x\to 0^+} -\frac{x}{\sqrt{x^2+0^2}} = \lim_{x\to 0^+} -1 = -1.$$

Since the limit is different along different paths, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

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51. Let $f(x,y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Find each of the following limits, or explain that the limit does not exist.

(a) $\lim_{(x,y)\to(0,1)} f(x,y)$

(b) $\lim_{(x,y)\to(2,3)} f(x,y)$

(c) $\lim_{(x,y)\to(0,0)} f(x,y)$

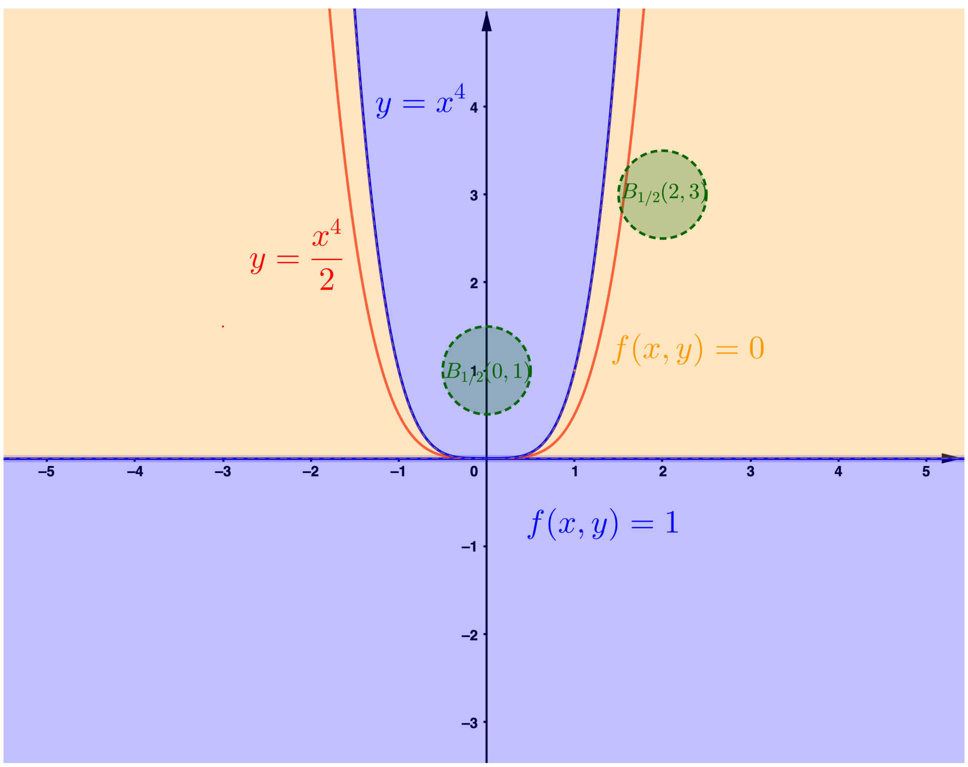
Solution. (a) Since any point (x,y) close enough to $(0,1)$ (for example $(x,y)\in B_{1/2}(0,1)$) satisfies $y\geq x^4$, we have

$$\lim_{(x,y)\to(0,1)} f(x,y) = \lim_{(x,y)\to(0,1)} 1 = 1.$$

(b) Since any point (x,y) close enough to $(2,3)$ (for example $(x,y)\in B_{1/2}(2,3)$) does not satisfy either $y\geq x^4$ or $y\leq 0$, we have

$$\lim_{(x,y)\to(2,3)} f(x,y) = \lim_{(x,y)\to(2,3)} 0 = 0.$$

(c) Note that $\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = 1$ while $\lim_{\substack{(x,y)\to(0,0)\\ y=x^4/2}} f(x,y) = 0$. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.



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55. Does knowing that

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy}?$$

Give reasons for your answer.

Solution. Note that $\displaystyle\lim_{(x,y)\to(0,0)} \left(1 - \frac{x^2y^2}{3}\right) = 1$ and $\displaystyle\lim_{(x,y)\to(0,0)} 1 = 1$.

By the Sandwich Theorem, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy} = 1$. ◀

61. Find the limit of $f(x,y) = \dfrac{x^3 - xy^2}{x^2+y^2}$ as $(x,y) \to (0,0)$ or show that the limit does not exist.

Solution. For $(x,y) \neq (0,0)$,

$$|f(x,y)| \leq \frac{x^2}{x^2+y^2} \cdot |x| + \frac{y^2}{x^2+y^2} \cdot |x| = |x|.$$

Since $\displaystyle\lim_{(x,y)\to(0,0)} |x| = 0$, it follows from the Sandwich Theorem that $\displaystyle\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$. So $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$. ◀

Exercise 14.3

8. Find $\partial f/\partial x$ and $\partial f/\partial y$ where $f(x,y) = \left(x^3 + (y/2)\right)^{2/3}$.

Solution. $\dfrac{\partial f}{\partial x} = \dfrac{2}{3}\left(x^3 + (y/2)\right)^{-1/3} \cdot 3x^2 = \dfrac{2x^2}{\sqrt[3]{x^3+(y/2)}}$,

$\dfrac{\partial f}{\partial y} = \dfrac{2}{3}\left(x^3 + (y/2)\right)^{-1/3} \cdot \dfrac{1}{2} = \dfrac{1}{3\sqrt[3]{x^3+(y/2)}}$. ◀

21. Find $\partial f/\partial x$ and $\partial f/\partial y$ where $f(x,y) = \displaystyle\int_x^y g(t)\,dt$ (g is continuous for all t).

Solution. By the Fundamental Theorem of Calculus, $\dfrac{\partial f}{\partial x} = -g(x)$ and $\dfrac{\partial f}{\partial y} = g(y)$. ◀

28. Find f_x, f_y and f_z where $f(x,y,z) = \sec^{-1}(x+yz)$.

Solution. Recall that $\dfrac{d\sec^{-1} t}{dt} = \dfrac{1}{|t|\sqrt{t^2-1}}$ for $|t| > 1$. Hence,

$$f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}, \quad f_y = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}, \quad f_z = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}.$$

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