MATH 2010E Advanced Calculus I
Suggested Solution of Homework 2

Exercise 11.3

68. **Vertical and horizontal lines**

    (a) Show that every every vertical lines in the xy-plane has a polar equation of the form $r = a\sec\theta$.

    (b) Find the analogous polar equation for horizontal lines in the xy-plane.

    Solution. (a) $x = a \implies r\cos\theta = a \implies r = a\sec\theta$.

    (b) $y = b \implies r\sin\theta = b \implies r = b\csc\theta$.

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Exercise 11.4

5. Identify the symmetries of the curves. Then sketch the curves in the xy-plane.

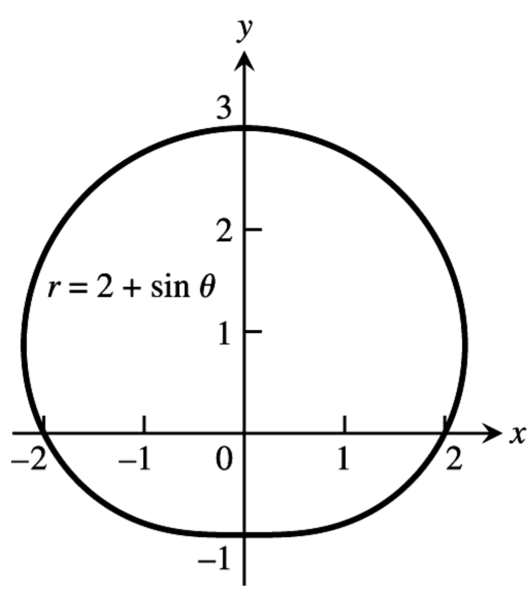
   $r = 2 + \sin\theta$.

   Solution. Suppose (r, θ) is an arbitrary point on the curve. Then $r = 2 + \sin\theta$.

   Since $2 + \sin(-\theta) = 2 - \sin\theta \neq r$ and $2 + \sin(\pi - \theta) = 2 + \sin\theta \neq -r$, the points $(r, -\theta)$, $(-r, \pi - \theta)$ do not lie on the curve. So the curve is not symmetric about the x-axis.

   Since $2 + \sin(\pi - \theta) = 2 + \sin\theta = r$, the point $(r, \pi - \theta)$ lies on the curve. So the curve is symmetric about the y-axis.

   Since the curve is symmetric about the y-axis but not the x-axis, it is not symmetric about the origin.



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Exercise 12.5

33. Find the distance from the point to the line.

$(0,0,12)$; $\quad x = 4t, \quad y = -2t, \quad z = 2t$

Solution. Write $S(0,0,12)$, and note that the line passes through $P(0,0,0)$ and is parallel to $\mathbf{v} := 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Then $\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 4 & -2 & 2 \end{vmatrix} = 24\mathbf{i} + 48\mathbf{j} = 24(\mathbf{i} + 2\mathbf{j})$.

So, the distance from the point S to the line is

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{24\sqrt{1^2 + 2^2 + 0^2}}{\sqrt{4^2 + 2^2 + 2^2}} = 2\sqrt{30}.$$

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47. Find the angles between the planes.

$x + y = 1, \quad 2x + y - 2z = 2$

Solution. Note that the angle between the planes is the angle between their normals $\mathbf{n}_1 := \mathbf{i} + \mathbf{j}$, $\mathbf{n}_2 := 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

So, the angle between the planes is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{2+1}{\sqrt{2}\sqrt{9}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

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54. Find the point in which the line meets the plane.

$x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$

Solution. Substitute the equations of the line into the equation of the plane, we have

$$\begin{aligned} 6(2) + 3(3 + 2t) - 4(-2 - 2t) &= -12 \\ 14t + 29 &= -12 \\ t &= -\frac{41}{14}. \end{aligned}$$

So $x = 2$, $y = 3 - \frac{41}{7}$, $z = -2 + \frac{41}{7}$. The point of intersection is $(2, -\frac{20}{7}, \frac{27}{7})$. ◀

Exercise 13.3

1. Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.
   $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}, \quad 0 \le t \le \pi$

   Solution. For $0 \le t \le \pi$,

$$\mathbf{r}'(t) = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \sqrt{5}\mathbf{k},$$

$$|\mathbf{r}'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (\sqrt{5})^2} = \sqrt{4(\sin^2 t + \cos^2 t) + 5} = 3.$$

   So the unit tangent vector is

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = (-\frac{2}{3}\sin t)\mathbf{i} + (\frac{2}{3}\cos t)\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k}, \quad 0 \le t \le \pi.$$

   Moreover, the required arclength of the curve is

$$\int_0^\pi |\mathbf{r}'(t)|\, dt = \int_0^\pi 3\, dt = 3\pi.$$

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Exercise 13.4

4. Find $\mathbf{T}$, $\mathbf{N}$ and κ for the plane curve.
   $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \quad t > 0$

   Solution. For $t > 0$,

$$\mathbf{r}'(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j},$$

$$|\mathbf{r}'(t)| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = |t| = t,$$

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},$$

$$\frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j},$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1,$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j},$$

$$\kappa = \frac{1}{|\mathbf{r}'|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t}.$$

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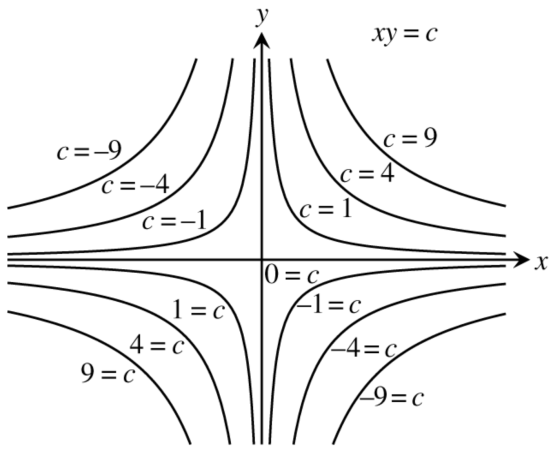
Exercise 14.1

15. Find and sketch the level curves $f(x,y) = c$ on the same set of coordinates axes for the given values of c.

    $f(x,y) = xy, \quad c = -9, -4, -1, 0, 1, 4, 9.$

    Solution.



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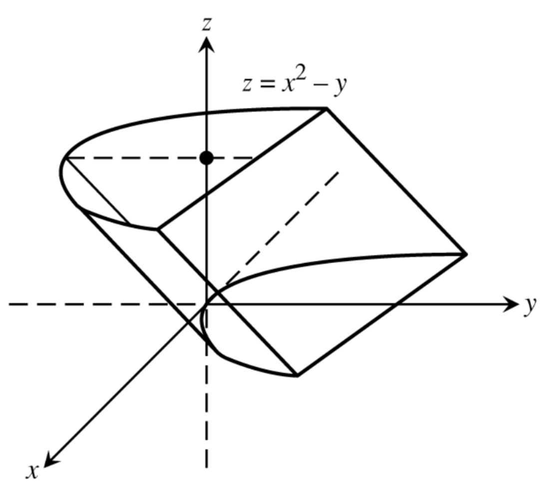
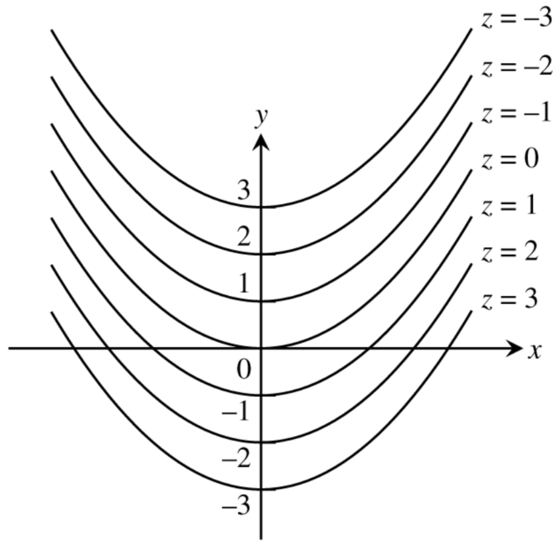
41. Display the values of the function $f(x,y) = x^2 - y$ in two ways: (a) by sketching the surface $z = f(x,y)$ and (b) by drawing an assortment of level curves in the function's domain. Label each curves with its function value.

    Solution.

    (a)



(b)



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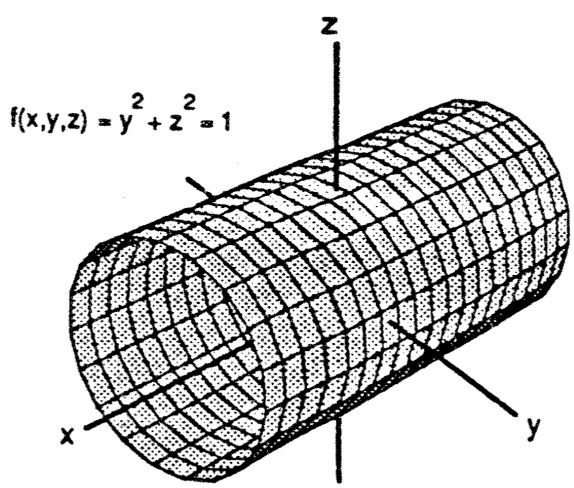
58. Sketch a typical level surface for the function $f(x, y, z) = y^2 + z^2$.

Solution.



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