MATH 2010E Advanced Calculus I
Suggested Solution of Homework 1

Exercise 12.3

5. $\mathbf{v} = 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Solution. (a) $\mathbf{v}\cdot\mathbf{u} = 0(1)+5(1)+(-3)(1) = 2$, $|\mathbf{v}| = \sqrt{0^2+5^2+(-3)^2} = \sqrt{34}$, $|\mathbf{u}| = \sqrt{3}$.

(b) Let θ be the angle between $\mathbf{v}$ and $\mathbf{u}$. Then $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{u}|} = \dfrac{2}{\sqrt{3}\sqrt{34}}$.

(c) The scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is $|\mathbf{u}|\cos\theta = \dfrac{2}{\sqrt{34}}$.

(d) $\text{proj}_{\mathbf{v}}\mathbf{u} = (|\mathbf{u}|\cos\theta)\dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{2}{\sqrt{34}}\cdot\dfrac{1}{\sqrt{34}}(5\mathbf{j}-3\mathbf{k}) = \dfrac{1}{17}(5\mathbf{j}-3\mathbf{k})$.

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14. Find the measures of the angles between the diagonals of the rectangle whose vertices are $A = (1, 0)$, $B = (0, 3)$, $C = (3, 4)$ and $D = (4, 1)$.

Solution. Note $\overrightarrow{AC} = \langle 2, 4\rangle$ and $\overrightarrow{BD} = \langle 4, -2\rangle$. Since $\overrightarrow{AC}\cdot\overrightarrow{BD} = 2(4) + 4(-2) = 0$, so angles measures are all $90°$. ◀

29. Using the definition of the projection of $\mathbf{u}$ onto $\mathbf{v}$, show by direct calculation that $(\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u})\cdot\text{proj}_{\mathbf{v}}\mathbf{u} = 0$.

Solution. Recall that $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$. Hence

$$
\begin{aligned}
(\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u})\cdot\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right)\cdot\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)(\mathbf{u}\cdot\mathbf{v}) - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)^2(\mathbf{v}\cdot\mathbf{v}) \\
&= \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{v}\|^2} - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{v}\|^2} = 0.
\end{aligned}
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Exercise 12.4

1. Find the length and direction (when defined) of $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$, where $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$.

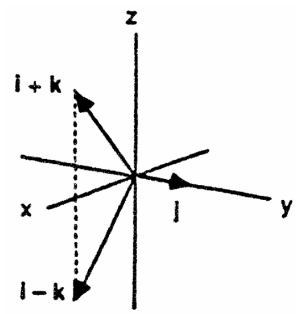
Solution. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Hence,

$$
\begin{aligned}
\text{length} &= \|\mathbf{u}\times\mathbf{v}\| = \sqrt{2^2+1^2+2^2} = 3, \\
\text{direction} &= \frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.
\end{aligned}
$$

Since $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v})$, so length $= 3$ and direction $= -\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$. ◀

10. Sketch the coordinate axes and then include the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ as vectors starting at the origin: $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j}$.

Solution. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{k}$.



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20. Verify that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ and find the volume of the parallelepiped (box) determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Solution. By direct calculations or properties of determinant, we have

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -2 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = 4.$$

That is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = 4.$$

So, volume of the parallelepiped $= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 4$. ◀

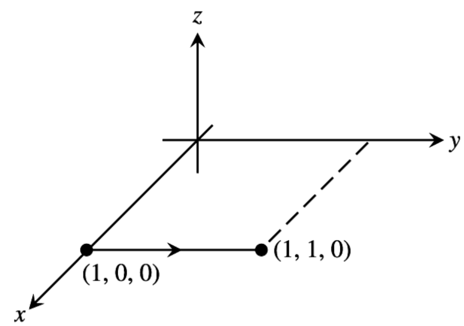
Exercise 12.5

2. Find a parametric equation for the line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$.

Solution. The direction is $\overrightarrow{PQ} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. A parametric equation for the required line is $(x, y, z) = (1, 2, -1) + t(-2, -2, 2)$. ◀

15. Find a parametrization for the line segments joining points $(1, 0, 0)$ and $(1, 1, 0)$. Draw coordinate axes and sketch each segment, indicating the direction of increasing t for your parametrization.

Solution. The direction is $\mathbf{j}$. A parametrization for the line segment is $(x, y, z) = (1, 0, 0) + t(0, 1, 0)$, where $0 \leq t \leq 1$.



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21. Find an equation of the plane through $P_0(0, 2, -1)$ and normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

Solution. An equation of the plane is

$$\begin{aligned} 3(x-0) + (-2)(y-2) + (-1)(z-(-1)) &= 0 \\ 3x - 2y - z &= -3. \end{aligned}$$

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23. Find an equation of the plane through $(1, 1, -1)$, $(2, 0, 2)$ and $(0, -2, 1)$.

Solution. Note that $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $-\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ are two vectors parallel to the plane. So

$$\mathbf{n} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$$

is normal to the plane. Hence, an equation of the plane is

$$\begin{aligned} 7(x-2) + (-5)(y-0) + (-4)(z-2) &= 0 \\ 7x - 5y - 4z &= 6. \end{aligned}$$

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