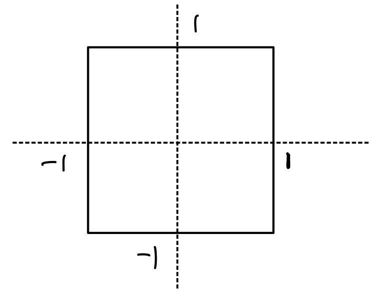
eg 2 Find global max and min of

$$f(x,y)=\sqrt{x^2+y^4}-y$$



on $R=\{(x,y)\in\mathbb{R}^2 : -1\leq x,y\leq 1\}$

Soln: (R is closed & bounded, f continuous $\Rightarrow \exists$ global max/min)

Step 1: critical points in $\operatorname{Int}(R)$

$$\overset{\shortparallel}{\{(x,y)\in\mathbb{R}^2 : -1< x,y< 1\}}$$

1st note that $\frac{\partial f}{\partial x}(0,0)$ does not exist (Ex!)

$$\Rightarrow \quad \vec{\nabla} f(0,0) \text{ DNE}$$

$\therefore$ (0,0) is a critical point.

Secondly, for $(x,y)\neq(0,0)$

$$\vec{\nabla} f=\left(\frac{x}{\sqrt{x^2+y^4}}, \frac{2y^3}{\sqrt{x^2+y^4}}-1\right)$$

$$\vec{0}=\vec{\nabla} f \Leftrightarrow \begin{cases} \dfrac{x}{\sqrt{x^2+y^4}}=0 \\ \dfrac{2y^3}{\sqrt{x^2+y^4}}-1=0 \end{cases}$$

$$\Leftrightarrow (x,y)=(0,\tfrac{1}{2})$$

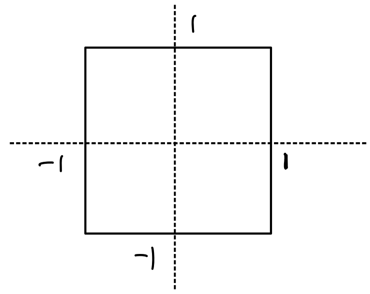
$\therefore (0,\frac{1}{2})$ is also a critical point.

Critical values: $f(0,0)=0$

$$f(0,\tfrac{1}{2})=\sqrt{0^2+(\tfrac{1}{2})^4}-\tfrac{1}{2}=-\tfrac{1}{4}$$

Step 2 Study boundary ∂R

$$\partial R = \{ x=1, -1 \le y \le 1 \} \cup \{ x=-1, -1 \le y \le 1 \} \cup \{ y=1, -1 \le x \le 1 \} \cup \{ y=-1, -1 \le x \le 1 \}$$



Case 1 $x = \pm 1,\ -1 \le y \le 1$.

$$f(\pm 1, y) = \sqrt{1+y^4} - y \Rightarrow 0 = 1 - 1 \le f \le \sqrt{2} + 1$$

Case 2 $y = 1,\ -1 \le x \le 1$

$$f(x,1) = \sqrt{x^2+1} - 1 \Rightarrow 0 \le f \le \sqrt{2} - 1$$

Case 3 $y = -1,\ -1 \le x \le 1$

$$f(x,-1) = \sqrt{x^2+1} + 1 \Rightarrow 2 \le f \le \sqrt{2} + 1$$

Hence on ∂R, f has min value 0 at $(0,1)$

max value $\sqrt{2}+1$ at $(\pm 1, -1)$

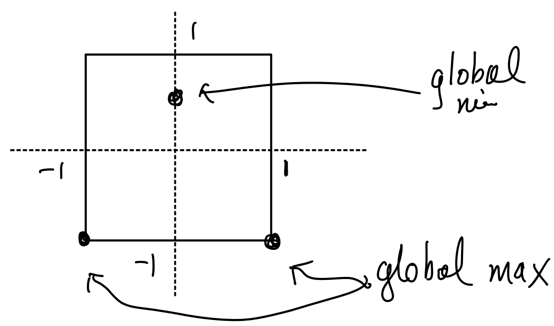
Step 3 Comparing values

(critical) $f(0,0) = 0$

$f(0, \frac{1}{2}) = -\frac{1}{4}$ $\leftarrow$ min

(boundary) $f(0,1) = 0$

$f(\pm 1, -1) = \sqrt{2} + 1$ $\leftarrow$ max



$\therefore$ f has global max at $(\pm 1, -1)$ with value $\sqrt{2}+1$

& has a global min at $(0, \frac{1}{2})$ with value $-\frac{1}{4}$

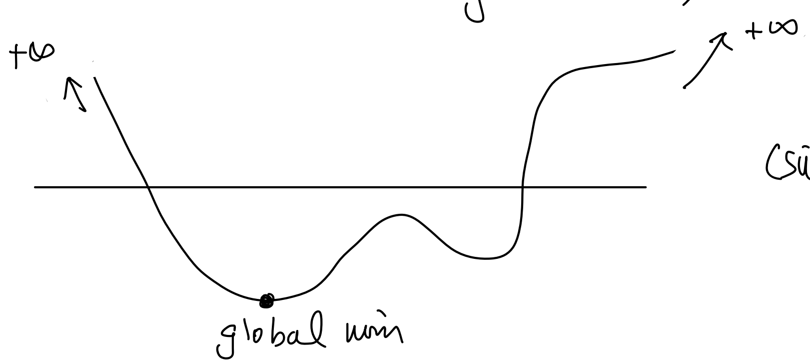
$\#$

Finding extrema on an unbounded region

Usually need conditions on $\lim_{\|\vec{x}\| \to +\infty} f(\vec{x})$ such as

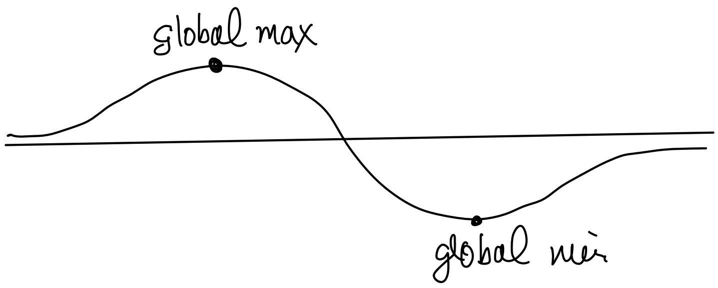
(1) $\lim_{\|\vec{x}\| \to +\infty} f(\vec{x}) = \pm\infty$ $\xrightarrow{+}$ No global max, Has global min

$\xrightarrow{-}$ Has global max, No global min,



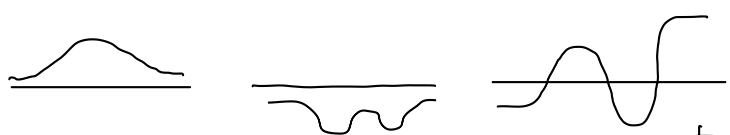
(similarly for $-\infty$)

(2) $\lim_{\|\vec{x}\| \to +\infty} f(\vec{x}) = C$ and $\exists\ \vec{a}, \vec{b} \in \mathbb{R}^n$ s.t. $f(\vec{a}) < C < f(\vec{b})$.



$\lim_{\|\vec{x}\| \to \infty} f(\vec{x}) = C$

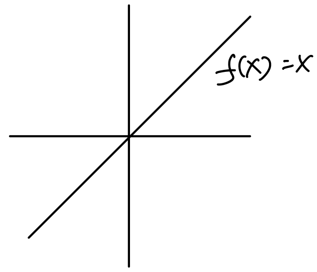
Remark: many other situations:



etc

Examples without global max or min:

(1) 1-variable $f(x) = x$ on $\mathbb{R}$



(2) 2-variables

$$g(x,y) = x^2 - y^2 - 4x + 6y + 7$$

$\lim\limits_{x \to \pm\infty} g(x,0) = +\infty \Rightarrow$ No global max

$\lim\limits_{y \to \pm\infty} g(0,y) = -\infty \Rightarrow$ No global min

$\left(\lim\limits_{\|(x,y)\| \to \infty} g(x,y) \text{ DNE} \right)$

eg 1 Find global extrema of

$$f(x,y) = x^2 + y^2 - 4x + 6y + 7 \text{ on } \mathbb{R}^2$$

Soln: $\lim\limits_{\|(x,y)\| \to \infty} f(x,y) = +\infty \Rightarrow$ No global max.

For global min, as $\operatorname{Int}(\mathbb{R}^2) = \mathbb{R}^2$, (no boundary) one only need to find critical points of f:

$$\vec{\nabla} f = \vec{0} \quad \text{or} \quad \underbrace{\vec{\nabla} f \text{ does not exist}}_{f \text{ polynomial, always exist}}$$

Since $\vec{\nabla} f = (2x - 4,\ 2y + 6) = (0,0)$

$\Leftrightarrow \quad x = 2 \ \& \ y = -3 \quad \therefore$ only critical pt is $(2,-3)$

$\therefore$ f has a global min at $(2,-3)$ with value $f(2,-3) = -6$

(check) ✗

eg 2 Find global extrema of

$$f(x,y) = y e^{-(x^2+y^2)} \quad \text{on } \mathbb{R}^2$$

Soln: Step 1 $\lim_{\|(x,y)\| \to +\infty} f(x,y) = 0$ (Ex!)

Step 2 Critical points

$$(0,0) = \vec{\nabla} f = \left(-2xy e^{-(x^2+y^2)},\ e^{-(x^2+y^2)} + (-2y) y e^{-(x^2+y^2)}\right)$$

$$= \left(-2xy e^{-(x^2+y^2)},\ (1-2y^2) e^{-(x^2+y^2)}\right)$$

$$\Leftrightarrow \begin{cases} xy = 0 \\ 1-2y^2 = 0 \end{cases} \Leftrightarrow x=0 \text{ \& } y = \pm \frac{1}{\sqrt{2}}$$

$\therefore$ Two critical points at $(0, \frac{1}{\sqrt{2}})$ and $(0, -\frac{1}{\sqrt{2}})$ with values

$$\begin{cases} f(0, \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}} \\ f(0, -\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}} e^{-\frac{1}{2}} \end{cases}$$

Step 3 Comparing values:

$$-\frac{1}{\sqrt{2}} e^{-\frac{1}{2}} < 0 < \frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$$

$\Rightarrow$ f has a global max at $(0, \frac{1}{\sqrt{2}})$ with value $\frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$ and a global min at $(0, \frac{-1}{\sqrt{2}})$ with value $\frac{-1}{\sqrt{2}} e^{-\frac{1}{2}}$

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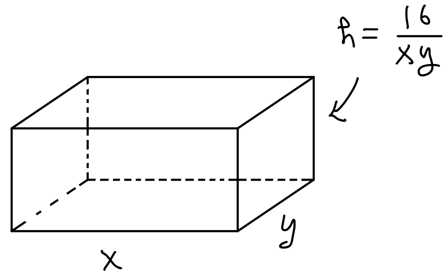
eg 3 Make a box, without top, with volume 16 unit

Cost : Base \$2 / unit area

Side \$0.5 / unit area

How to minimize cost?



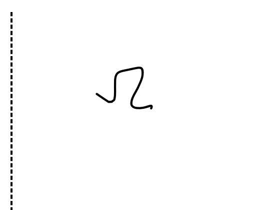
Soln Minimize cost

$$C(x,y) = \underbrace{2}_{\text{cost}} \cdot \underbrace{xy}_{\text{base}} + \underbrace{0.5}_{\text{cost}} \cdot \Big(2 \cdot \underbrace{\frac{16}{xy} x}_{\substack{\text{sides} \\ \text{front \& back}}} + 2 \cdot \underbrace{\frac{16}{xy} y}_{\substack{\text{sides} \\ \text{left \& right}}} \Big)$$

$$= 2xy + \frac{16}{x} + \frac{16}{y}$$

Domain $\Omega = \{(x,y) \in \mathbb{R}^2 : x, y > 0\}$ (Physical dimensions > 0)

(neither closed nor bounded)



Step 1: Critical point(s)

$C(x,y)$ rational function & $x \neq 0$, $y \neq 0$ on Ω

$\Rightarrow$ $\vec{\nabla} C$ always exist

& $\vec{\nabla} C = \left(2y - \frac{16}{x^2}, \ 2x - \frac{16}{y^2}\right)$

$$\vec{\nabla} C = \vec{0} \iff \begin{cases} 2y - \frac{16}{x^2} = 0 \\ 2x - \frac{16}{y^2} = 0 \end{cases}$$

(together with $x>0, y>0$) $\Leftrightarrow$ $(x,y)=(2,2)$ (Check!)

Only one critical pt. $(2,2)$

& $C(2,2)=24$.

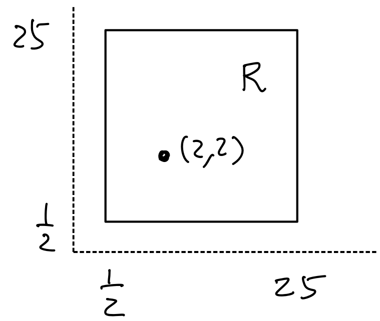
<u>Step 2</u>: Method 1: Show that

$$\lim_{\substack{\|(x,y)\|\to+\infty \\ (x,y)\in\Omega}} C(x,y) = +\infty = \lim_{(x,y)\to\partial\Omega} C(x,y) \quad \text{(Ex!)}$$

<u>Method 2</u> (As we don't care about global max)

Find rectangle R containing $(2,2)$ such that

$C(x,y) > 24$ on ∂R and outside R



A possible choice of R is

$$R = \{(x,y): \tfrac{1}{2} \le x \le 25,\ \tfrac{1}{2} \le y \le 25\}$$

Then for points on ∂R or outside R,

we have either $0 < x \le \frac{1}{2}$, $0 < y \le \frac{1}{2}$,

$\{(x,y): x \ge \frac{1}{2}, y \ge 25\}$, or $\{(x,y): x \ge 25, y \ge \frac{1}{2}\}$

<u>Case 1</u>, For $0 < x \le \frac{1}{2}$,

$$C(x,y) > \frac{16}{x} \ge 32 > 24$$

Case 2: For $0 < y \leq \frac{1}{2}$,

$$C(x,y) > \frac{16}{y} \geq 32 > 24$$

Case 3 $x \geq \frac{1}{2}$, $y \geq 25$

$$C(x,y) > 2xy \geq 2 \cdot \frac{1}{2} \cdot 25 = 25 > 24$$

Case 4 $x \geq 25$, $y \geq \frac{1}{2}$

$$C(x,y) > 2xy \geq 25 > 24 .$$

Hence $C(x,y)$ has a global min at $(2,2)$ with value 24.

※

Taylor Series Expansion

Recall: 1-variable Taylor expansion: (at $t = t_0$)

$$g(t) = g(t_0) + g'(t_0)(t-t_0) + \frac{1}{2!} g''(t_0)(t-t_0)^2 + \cdots + \frac{1}{k!} g^{(k)}(t_0)(t-t_0)^k + \text{remainder}.$$

For several variables, we have

Thm (Taylor's Thm)

Let $\begin{cases} \bullet\ f: \Omega \to \mathbb{R},\ (\Omega \subseteq \mathbb{R}^n, \text{ open}) \text{ be } C^k, \\ \bullet\ \vec{a} \in \Omega, \end{cases}$

Then for any $\vec{x} \in \Omega$ "near" $\vec{a}$,

$$\boxed{\begin{aligned} f(\vec{x}) = f(\vec{a}) &+ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \frac{1}{2!}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})(x_i-a_i)(x_j-a_j) \\ &+ \cdots + \frac{1}{k!}\sum_{i_1,\dots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1}\cdots \partial x_{i_k}}(\vec{a})(x_{i_1}-a_{i_1})\cdots(x_{i_k}-a_{i_k}) + \mathcal{E}_k(\vec{x},\vec{a}) \end{aligned}}$$

with $\displaystyle \lim_{\vec{x}\to\vec{a}} \frac{\mathcal{E}_k(x,a)}{\|\vec{x}-\vec{a}\|^k} = 0$

In particular, for C^2 function of 2-variables

$$\boxed{\begin{aligned} f(x,y) = f(a,b) &+ \left(\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)\right) \\ &+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2\right) \\ &+ \mathcal{E}_2(\vec{x}) \end{aligned}}$$

(used Clairaut's Thm)

Def:

$$P_k(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \frac{1}{2!}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})(x_i-a_i)(x_j-a_j)$$

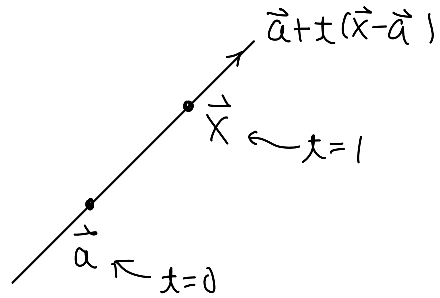
$$+ \cdots + \frac{1}{k!}\sum_{i_1,\dots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(\vec{a})(x_{i_1}-a_{i_1})\cdots(x_{i_k}-a_{i_k})$$

is called the <u>k-th order Taylor polynomial</u> of f at $\vec{a}$

<u>Remarks</u>: (1) $P_1(\vec{x}) = L(\vec{x})$ the Linearization of f at $\vec{a}$

(2) P_k & f have <u>equal</u> partial derivatives up to order k at $\vec{a}$.

<u>Idea of proof</u>:



$\vec{x} \in \Omega$ near $\vec{a}$, define

$$g(t) = f(\vec{a}+t(\vec{x}-\vec{a}))$$

and use 1-variable Taylor's expansion.

$$f(\vec{x}) = g(1) = g(0) + g'(0) + \frac{1}{2}g''(0) + \cdots + \frac{1}{k!}g^{(k)}(0) + \varepsilon_k(\overset{t=1}{1})$$

$$g(0) = f(\vec{a}+0\cdot(\vec{x}-\vec{a})) = f(\vec{a})$$

$$g'(t) = \vec{\nabla} f(\vec{a}+t(\vec{x}-\vec{a})) \cdot \frac{d}{dt}(\vec{a}+t(\vec{x}-\vec{a}))$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a}+t(\vec{x}-\vec{a}))(x_i - a_i)$$

$$\Rightarrow \quad g'(0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i)$$

$$g''(t) = \frac{d}{dt} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (\vec{a} + t(\vec{x} - \vec{a}))(x_i - a_i)$$

$$= \sum_{i=1}^{n} \frac{d}{dt} \left[\frac{\partial f}{\partial x_i} (\vec{a} + t(\vec{x} - \vec{a})) \right] (x_i - a_i)$$

$$= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} (\vec{a} + t(\vec{x} - \vec{a})) \right) (x_j - a_j) \right] (x_i - a_i)$$

$$\therefore \quad g''(0) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} (\vec{a}) (x_i - a_i)(x_j - a_j)$$

$$\vdots$$

And so on, one can prove the formula for Taylor's Thm. (Detail omitted)

eg1 $f(x,y) = e^x \cos y$

Find 2^{nd} order Taylor polynomial at $\vec{a} = (0,0)$

Soln: $f_x = e^x \cos y \qquad f_y = -e^x \sin y$

$f_{xx} = e^x \cos y \qquad f_{xy} = -e^x \sin y = f_{yx}$

$f_{yy} = -e^x \cos y$

$\Rightarrow \quad f_x(0,0) = 1, \quad f_y(0,0) = 0$

$f_{xx}(0,0) = 1, \quad f_{xy}(0,0) = f_{yx}(0,0) = 0$

$f_{yy}(0,0) = -1$

Also $f(0,0)=1$

$\therefore$ Taylor polynomial of order of f at $(0,0)$ is

$$P_2(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\left[f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right]$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2$$

(Ex!) Find $P_3(x,y)$ at $(0,0)$.

Answer: $P_3(x,y) = 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{6}x^3 - \frac{1}{2}xy^2$

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