eg3
$$\begin{cases} x^2+y^2+z^2=2 \\ x+z=1 \end{cases}$$
Solve for $y=y(x)$, $z=z(x)$? near $(0,1,1)$

(x, y, z) $\longleftrightarrow$ (x, y_1, y_2)

$\mathbb{R}^3$ notation	General Notation
$g(x,y,z) = x^2+y^2+z^2=2$	$F_1(x, y_1, y_2) = x_1^2 + y_1^2 + y_2^2 = c_1$
$h(x,y,z) = x+z = 1$	$F_2(x, y_1, y_2) = x_1 + y_2 = c_2$
near $(0,1,1)$	$a=0$, $\vec{b}=(b_1,b_2)=(1,1)$ $\vec{c}=(c_1,c_2)=(2,1)$
$\begin{bmatrix} \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}_{(0,1,1)} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ invertiable (det $\neq 0$)	$\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}_{(a,b_1,b_2)} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ invertiable (det $\neq 0$)

By IFT

$\exists$ $y=y(x)$, $z=z(x)$ near $(0,1,1)$ such that
$$\begin{cases} g(x,y(x),z(x)) = 2 \\ h(x,y(x),z(x)) = 1 \\ y(0)=1 \\ z(0)=1 \end{cases}$$

Remark: $\frac{dy}{dx}, \frac{dz}{dx}\Big|_{x=0}$ can be calculated by Implicit Diff.

By IFT

$\exists$ $y_1=\varphi_1(x)$, $y_2=\varphi_2(x)$ near (a,b_1,b_2) such that
$$\begin{cases} F_1(x,\varphi_1(x),\varphi_2(x)) = c_1 \\ F_2(x,\varphi_1(x),\varphi_2(x)) = c_2 \\ \varphi_1(a)=b_1 \\ \varphi_2(a)=b_2 \end{cases}$$

Remark $\frac{d\varphi_1}{dx}, \frac{d\varphi_2}{dx}\Big|_{x=a}$ can be calculated by Implicit Diff.

eg: Consider the constraints

$$\begin{cases} xz + \sin(yz - x^2) = 8 \\ x + 4y + 3z \quad = 18 \end{cases}$$

$(2,1,4)$ is a solution.

Can we solve 2 of the variables as functions of the remaining variable?

Soln:

$$\vec{F}(x,y,z) = \begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \end{bmatrix} = \begin{bmatrix} xz + \sin(yz - x^2) \\ x + 4y + 3z \end{bmatrix}$$

$$D\vec{F} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} z - 2x\cos(yz - x^2) & z\cos(yz - x^2) & x + y\cos(yz - x^2) \\ 1 & 4 & 3 \end{bmatrix}$$

$$\Rightarrow D\vec{F}(2,1,4) = \begin{bmatrix} 0 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix} \qquad \text{(check!)}$$

$$\begin{array}{ccc} & x & y \\ & \downarrow & \downarrow \\ \det & \multicolumn{2}{c}{\begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}} \end{array} \qquad \begin{array}{ccc} & x & z \\ & \downarrow & \downarrow \\ \det & \multicolumn{2}{c}{\begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix}} \end{array} \qquad \begin{array}{ccc} & y & z \\ & \downarrow & \downarrow \\ \det & \multicolumn{2}{c}{\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}} \end{array}$$

$$= -4 \neq 0 \qquad\qquad = -3 \neq 0 \qquad\qquad = 0$$

IFT $\Rightarrow$ • x, y can be solved as (diff) functions of z near $(2, 1, 4)$

• x, z can be solved as (diff) functions of y near $(2, 1, 4)$

• <u>No conclusion</u> on whether y, z can be solved as (diff) functions of x near $(2, 1, 4)$.

<u>Remark</u>: Implicit diff. $\Rightarrow$ (if $y = y(x)$, $z = z(x)$ diff.)

$$\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which has no solution, a contradiction!

Thm (Inverse Function Theorem)

Let $\vec{f} : \Omega \to \mathbb{R}^n$ be C^1, ($\Omega \subset \mathbb{R}^n$, open)

Suppose $D\vec{f}(\vec{a})$ is invertible ($n \times n$ matrix)

Then $\exists$ open sets $U \subseteq \mathbb{R}^n$ containing $\vec{a}$,
$V \subseteq \mathbb{R}^n$ containing $\vec{b} = \vec{f}(\vec{a})$

such that $\exists$ a unique function

$$\vec{g} : V \to U \quad \text{with}$$

$$\vec{g}(\vec{b}) = \vec{a}$$

satisfying $\begin{cases} \vec{g}(\vec{f}(\vec{x})) = \vec{x}, & \forall \vec{x} \in U \\ \vec{f}(\vec{g}(\vec{y})) = \vec{y}, & \forall \vec{y} \in V \end{cases}$ (i.e. $\vec{g} = (\vec{f}|_U)^{-1}$)

Moreover, $\vec{g}$ is C^1 and

$$D\vec{g}(\vec{y}) = [D\vec{f}(\vec{g}(\vec{y}))]^{-1}, \quad \forall \vec{y} \in V.$$

(Pf: MATH3060)

Remark: $\vec{g} = (\vec{f}|_U)^{-1}$ is called a local inverse of $\vec{f}$ at $\vec{a}$.

eg: $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \vec{f}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$

Clearly, $\vec{f}$ is not globally invertible: $\vec{f}\begin{bmatrix} -x \\ -y \end{bmatrix} = \vec{f}\begin{bmatrix} x \\ y \end{bmatrix}$
(2 to 1)

Local inverse?

To check this:

$$D\vec{f} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$\det D\vec{f} = 4x^2+4y^2 \geq 0 \quad \text{\& "}=0 \Leftrightarrow (x,y)=(0,0)\text{"}.$$

For $(x,y) \neq (0,0)$, IFT (Inverse Function Thm)

$\vec{f}$ has a locally inverse at (x,y) $(\neq (0,0))$.

For instance, let $(x,y) = (1,-1)$ &

$\vec{g}(u,v)$ be a local inverse of $f(x,y)$

near $(x,y) = (1,-1)$.

$\left(\text{where } \begin{matrix} u = x^2-y^2 \\ v = 2xy \end{matrix} \right)$

$$\vec{f}(1,-1) = (0,-2) \Rightarrow g(0,-2) = (1,-1)$$

$$D\vec{g}(0,-2) = \left(Df(1,-1)\right)^{-1} = \left(\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}_{(1,-1)} \right)^{-1}$$

$$= \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{(check!)}$$

<u>Explicit calculation of $\vec{g}(u,v)$:</u>

$$\begin{cases} u = x^2-y^2 \\ v = 2xy \end{cases}$$

near $(x,y) = (1,-1) \Rightarrow x \neq 0 \Rightarrow y = \frac{v}{2x}$

$$\Rightarrow \quad u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$\Rightarrow \quad 4x^4 - 4ux^2 - v^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{4u \pm \sqrt{(-4u)^2 - 4\cdot 4(-v^2)}}{8}$$

$$= \frac{u \pm \sqrt{u^2+v^2}}{2}$$

Put $(x,y) = (1,-1) \Rightarrow (u,v) = (0,-2)$

$$1^2 = \frac{0 \pm \sqrt{0^2+(-2)^2}}{2}$$

$\Rightarrow$ "$-$" should be rejected !

$$\Rightarrow \quad x^2 = \frac{u+\sqrt{u^2+v^2}}{2}$$

$$\Rightarrow \quad x = \sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}$$

("$-$" rejected as x near 1)

& hence $y = \dfrac{2v}{x} = \dfrac{2\sqrt{2}\,v}{\sqrt{u+\sqrt{u^2+v^2}}}$

$$\therefore \quad g(u,v) = \left(\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}},\ \frac{2\sqrt{2}\,v}{\sqrt{u+\sqrt{u^2+v^2}}} \right)$$

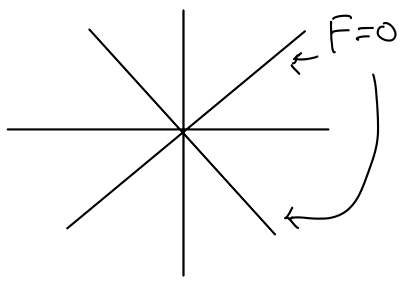
near $(0,-2)$

Remark: In Implicit Function Thm & Inverse Function Thm, we need to check det. of Jacobian matrix (a submatrix) is nonzero. In case that the det $= 0$, we have No conclusion:

eg: Implicit Function Thm

$F(x,y) = x^2 - y^2 = 0$

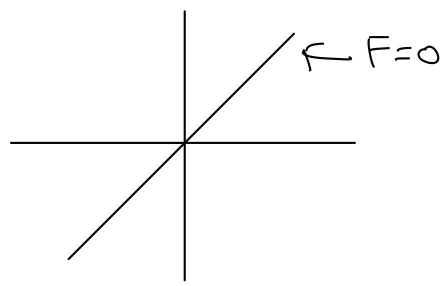
$$\boxed{\frac{\partial F}{\partial y} = -2y \Big|_{(0,0)} = 0}$$



y is not locally a function of x near $(0,0)$

$F(x,y) = x^3 - y^3 = 0$

$$\boxed{\frac{\partial F}{\partial y} = -3y^2 \Big|_{(0,0)} = 0}$$

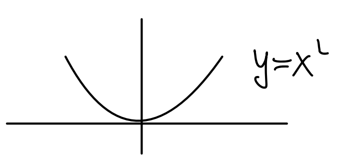


$y = x$ is a function of x (both globally & locally) near $(0,0)$

Inverse function Thm

$f(x) = x^2$

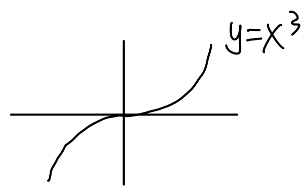
$$\boxed{f'(0) = 0}$$



Not injective near $x=0$
$\Rightarrow$ no local inverse near $x=0$

$f(x) = x^3$

$$\boxed{f'(0) = 0}$$



$g(y) = y^{1/3}$ is a local inverse near $x=0$ (in fact, also global)

Brief review

Basic geometry : vectors, lines, planes,
curves (tangent vectors, arc-length),
open set, closed set,
interior, exterior, boundary

Limit : Definition, Squeeze Thm, Continuity

Partial derivative : 1^{st} and higher order,
Clairaut's Thm (Mixed derivatives thm)
C^k- functions

---
mid-term

Differentiability : Linearization, gradient,
directional derivative, total differential

Chain Rule : Jacobian Matrix, normal vector to level set,
Implicit differentiation

Extremum : global max/min on closed & bounded set

Taylor's expansion : 2^{nd} derivative test,
Classification of local extremum

Lagrange Multiplier: Constrainted problem, Quadratic constraints

Implicit Function Theorem & Inverse Function Theorem