Implicit Function Theorem

Recall: Implicit differentiation

eg. $x^2+y^2+z^2=2$ and if $z=z(x,y)$,

then

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)=0 \Rightarrow 2x+2z\frac{\partial z}{\partial x}=0$$

$$\frac{\partial}{\partial y}(x^2+y^2+z^2)=0 \Rightarrow 2y+2z\frac{\partial z}{\partial y}=0$$

If the point (x,y,z) satisfies $z\neq 0$

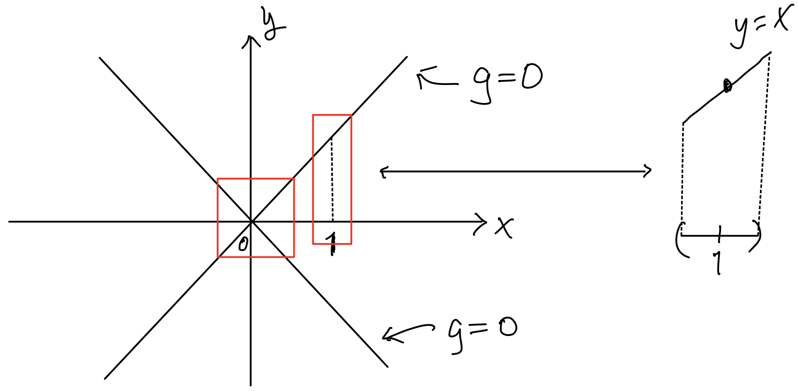
then we have $\frac{\partial z}{\partial x}=-\frac{x}{z}$ & $\frac{\partial z}{\partial y}=-\frac{y}{z}$

Question: If a level set $g(x,y)=c$ (or more generally)
is given, can we "solve" the constraint?

i.e. can we find $y=h(x)$ s.t. $g(x,h(x))=c$

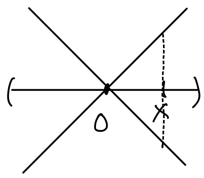
or $x=k(y)$ s.t. $g(k(y),y)=c$?

eg1 $g(x,y)=x^2-y^2=0$ $(\Rightarrow x=\pm y)$



Yes, we can solve for $y=h(x)$ near $(x,y)=(1,1)$

But near $(x,y)=(0,0)$



2 y-values $\longleftrightarrow$ 1 x-value

It cannot be a graph of a function $y=h(x)$

$\Rightarrow$ Cannot solve for a function $y=h(x)$ near $(0,0)$.

<u>eg2</u> $S: x^2+y^2+z^2=2$ in $\mathbb{R}^3$

Can we solve $z=h(x,y)$ near $(0,1,1)$?

Can we solve $x=k(y,z)$ near $(0,1,1)$?

<u>Observations:</u>

1st question: if $z=h(x,y)$ exists, then

$$\begin{cases} \partial_x(x^2+y^2+z^2)=0 \\ \partial_y(x^2+y^2+z^2)=0 \end{cases} \Rightarrow \begin{cases} \frac{\partial z}{\partial x} = -\frac{x}{z} \\ \frac{\partial z}{\partial y} = -\frac{y}{z} \end{cases} \quad \text{provided that } z\neq 0.$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}(0,1,1)=0, \quad \frac{\partial z}{\partial y}(0,1,1)=-1$$

At least, there is no contradiction & we have a hope to solve it!

2nd question: if $x=k(y,z)$ exists

$$\begin{cases} \partial_y(x^2+y^2+z^2)=0 \\ \partial_z(x^2+y^2+z^2)=0 \end{cases} \Rightarrow \begin{cases} 2x\frac{\partial x}{\partial y}+2y=0 \\ 2x\frac{\partial x}{\partial z}+2z=0 \end{cases}$$

At the point $(0,1,1)$, we have $\begin{cases} 0+2=0 \\ 0+2=0 \end{cases}$

which is a contradiction.

So there exists <u>NO</u> $x = h(y,z)$ (which is differentible)

at near the point $(x,y,z) = (0,1,1)$.

<u>General situation (in 3-variables)</u>

$$F(x,y,z) = c$$

If $z = z(x,y)$ (differentiable), then implicit differentiation

$$\begin{array}{ll} \dfrac{\partial}{\partial x}: & \left\{ \begin{array}{l} \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x} = 0 \\[2ex] \dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial y} = 0 \end{array} \right. \\ \dfrac{\partial}{\partial y}: & \end{array}$$

If $F(\vec{a}) = c$ & $\underline{\underline{\dfrac{\partial F}{\partial z}(\vec{a}) \neq 0}}$, then

$$\begin{bmatrix} \dfrac{\partial z}{\partial x} \\[2ex] \dfrac{\partial z}{\partial y} \end{bmatrix} = -\frac{1}{\dfrac{\partial F}{\partial z}(\vec{a})} \begin{bmatrix} \dfrac{\partial F}{\partial x}(\vec{a}) \\[2ex] \dfrac{\partial F}{\partial y}(\vec{a}) \end{bmatrix}$$

<u>eg3</u> (Multiple constraints)

$$\mathcal{C} \left\{ \begin{array}{l} x^2+y^2+z^2 = 2 \\ x+z = 1 \end{array} \right. \qquad \left(\begin{array}{l} \text{3-variables, 2 equations} \\ \text{expect } \mathcal{C} \text{ is "1-dim"} \end{array} \right)$$

<u>Question</u>: Can we solve $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}$?

<u>Observation</u>: If we have $y = y(x)$ & $z = z(x)$, differentiable

then $$\frac{d}{dx}\left(x^2 + (y(x))^2 + (z(x))^2\right) = 0$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} + 2z\frac{dz}{dx} = 0$$

$$\Rightarrow \quad y\frac{dy}{dx} + z\frac{dz}{dx} = -x \qquad (1)$$

and $$\frac{d}{dx}(x + z(x)) = 0$$

$$\Rightarrow \quad 1 + \frac{dz}{dx} = 0 \qquad (2)$$

$$\therefore \quad \begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} -x \\ -1 \end{bmatrix}$$

If $\underline{\underline{\det \begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix}}} \neq 0$, then one can solve (uniquely) for $\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix}$.

So we have a hope to the existence of $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}$.

For instance $(x, y, z) = (0, 1, 1)$ (on $\mathcal{C}$)

$$\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{is solvable}$$

and $$\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -1 \end{bmatrix} \overset{\text{(check)}}{=} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

In general, given $\mathcal{C}: \begin{cases} F_1(x,y,z) = C_1 \\ F_2(x,y,z) = C_2 \end{cases}$

$$\left(\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad \vec{F}(\vec{x}) = \vec{C}, \quad \vec{F}: \mathbb{R}^3 \to \mathbb{R}^2 \right)$$

Suppose $F_i(a,b,c) = C_i$, $i = 1, 2$

Assume $y = y(x)$, $z = z(x)$ near (a,b,c) (diff.)

(Implicit differentiation)

$$\begin{cases} \frac{d}{dx} F_1(x, y(x), z(x)) = 0 \\ \frac{d}{dx} F_2(x, y(x), z(x)) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}\frac{dy}{dx} + \frac{\partial F_1}{\partial z}\frac{dz}{dx} = 0 \\ \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}\frac{dy}{dx} + \frac{\partial F_2}{\partial z}\frac{dz}{dx} = 0 \end{cases}$$

i.e. $$\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{bmatrix}$$

$\therefore$ If $\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}$ at (a,b,c) is <u>invertiable</u> (i.e. $\det(\;) \neq 0$)

then $$\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{bmatrix} \quad (\text{at } (a,b,c))$$

General dimensions

Given $n+k$ variables, k equations

$$(x_1, \cdots x_n, y_1, \cdots, y_k) \quad n+k \text{ variables}$$

$$\begin{cases} F_1(x_1, \cdots, x_n, y_1, \cdots, y_k) = c_1 \\ \vdots \\ F_k(x_1, \cdots, x_n, y_1, \cdots, y_k) = c_k \end{cases}$$

expect: $y_1, \cdots, y_k$ can be solved as functions of $x_1, \cdots, x_n$.

Thm (Implicit Function Theorem)

Let $\Omega \subseteq \mathbb{R}^{n+k}$ be open, $\vec{F}: \Omega \to \mathbb{R}^k$, $\vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_k \end{bmatrix}$ be C^1

Denote $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ & $\vec{y} = (y_1, \cdots, y_k) \in \mathbb{R}^k$.

$$\vec{F}(\vec{x}, \vec{y}) = \begin{bmatrix} F_1(\vec{x}, \vec{y}) \\ \vdots \\ F_k(\vec{x}, \vec{y}) \end{bmatrix} = \begin{bmatrix} F_1(x_1, \cdots, x_n, y_1, \cdots, y_k) \\ \vdots \\ F_k(x_1, \cdots, x_n, y_1, \cdots, y_k) \end{bmatrix}$$

Suppose $(\vec{a}, \vec{b}) \in \Omega$, where $\vec{a} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^k$ such that

$$\vec{F}(\vec{a}, \vec{b}) = \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$$

and the $k \times k$ matrix

$$\left[\frac{\partial F_i}{\partial y_j}(\vec{a}, \vec{b}) \right]_{1 \le i, j \le k} = \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(\vec{a}, \vec{b}) & \cdots & \frac{\partial F_1}{\partial y_k}(\vec{a}, \vec{b}) \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial y_1}(\vec{a}, \vec{b}) & \cdots & \frac{\partial F_k}{\partial y_k}(\vec{a}, \vec{b}) \end{bmatrix}$$

is invertible (i.e. $\det \left[\frac{\partial F_i}{\partial y_j}(\vec{a}, \vec{b}) \right] \neq 0$)

Then there are open sets $U \subseteq \mathbb{R}^n$ containing $\vec{a}$, and $V \subseteq \mathbb{R}^k$ containing $\vec{b}$ such that there exists a unique function $\vec{\varphi}: U \to V$ with $\vec{\varphi}(\vec{a}) = \vec{b}$ and

$$\vec{F}(\vec{x}, \vec{\varphi}(\vec{x})) = \vec{c}, \quad \forall \vec{x} \in U$$

Moreover, $\vec{\varphi}$ is C^1 and (by implicit differentiation)

$$\left[\frac{\partial \varphi_j}{\partial x_l}(\vec{x}) \right]_{k \times n} = - \left[\frac{\partial F_i}{\partial y_j}(\vec{x}, \vec{\varphi}(\vec{x})) \right]_{k \times k}^{-1} \left[\frac{\partial F_i}{\partial x_l}(\vec{x}, \vec{\varphi}(\vec{x})) \right]_{k \times n}$$

(Pf: in MATH3060)

<u>Special case</u> (A): $k=1$ (1 constraint)

$F: \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}$

$F(\vec{x}, \vec{y}) = F(x_1, \cdots, x_n, y) = c$ (Constraint)

Suppose $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$, $b \in \mathbb{R}$ s.t.

$F(a_1, \cdots, a_n, b) = c$

<u>IFT</u>: If $\frac{\partial F}{\partial y}(a_1, \cdots, a_n, b) \neq 0$, then

$\exists\ U^{\text{open}}$ s.t. $(a_1, \cdots, a_n) \in U$ and $V^{\text{open}} \subset \mathbb{R}$ s.t. $b \in V$ (interval)

& $\varphi: U \to V$ s.t. $\varphi(a_1, \cdots, a_n) = b$ &

$F(x_1, \cdots, x_n, \varphi(x_1, \cdots, x_n)) = c \quad \forall (x_1, \cdots, x_n) \in U$

i.e. $y = \varphi(x_1, \cdots, x_n)$ "near" $(a_1, \cdots, a_n)$ solving the constraint $F(x_1, \cdots, x_n, y) = c$

(at the point $y(x_1, \cdots, x_n) = b$)

In eg2: $x^2+y^2+z^2=2$ Solve $z=z(x,y)$

(x,y,z) $\mathbb{R}^3$ notation	(x_1,x_2,y) general notation (this "y" is not the "y" on the other side)
$g(x,y,z)=x^2+y^2+z^2=2$ near $(0,1,1)$	$F(x_1,x_2,y)=x_1^2+x_2^2+y^2=c$ $(c=2)$ $\vec{a}=(a_1,a_2)=(0,1)$, $b=1$
$\frac{\partial g}{\partial z}(0,1,1)=2\neq 0$ By IFT $\exists\, z=z(x,y)$ "near" $(0,1,1)$ s.t. $\begin{cases} g(x,y,z(x,y))=2 \\ z(0,1)=1 \end{cases}$	$\frac{\partial F}{\partial y}(a_1,a_2,b)=2\neq 0$ By IFT $\exists\, y=\varphi(x_1,x_2)$ "near" (a_1,a_2,b) s.t. $\begin{cases} F(x_1,x_2,\varphi(x_1,x_2))=c \\ \varphi(a_1,a_2)=b \end{cases}$ $(\varphi(0,1)=1)$
$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ can be computed by implicit differentiation.	$\frac{\partial \varphi}{\partial x_1}$, $\frac{\partial \varphi}{\partial x_2}$ can be computed by implicit differentiation.

Special case (B) $\quad n=1, k=2 \quad$ (2-constraints)

$$\vec{F}: \Omega \subseteq \mathbb{R}^{1+2} \longrightarrow \mathbb{R}^2$$

$$\vec{F}(x, y_1, y_2) = \begin{bmatrix} F_1(x, y_1, y_2) \\ F_2(x, y_1, y_2) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Suppose (a, b_1, b_2) satisfies the constraints $\vec{F}(a, b_1, b_2) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

then IFT means

If $\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}(a, b_1, b_2)$ is invertible $(\det \neq 0)$

then $\exists\ y_1 = \varphi_1(x)$ & $y_2 = \varphi_2(x)$ "near" (a, b_1, b_2)

solving the constraints (locally)

$$\begin{cases} F_1(x, \varphi_1(x), \varphi_2(x)) = c_1 \\ F_2(x, \varphi_1(x), \varphi_2(x)) = c_2 \end{cases}$$

& $\quad (\varphi_1(a), \varphi_2(a)) = (b_1, b_2)$.