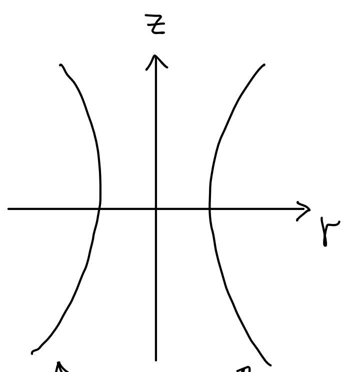
eg2 $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Up to scaling of each variables, graph looks like graph of

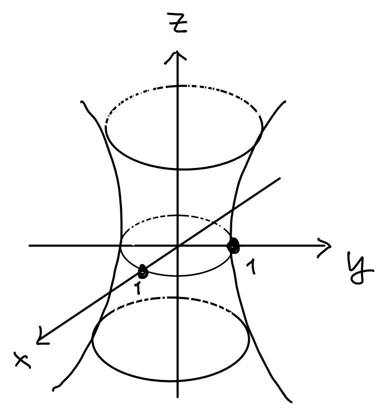
$$x^2+y^2-z^2=1$$

Using polar coordinates on xy-plane, the constraint can be written as $\quad r^2 - z^2 = 1 \quad$, where $r^2 = x^2+y^2$

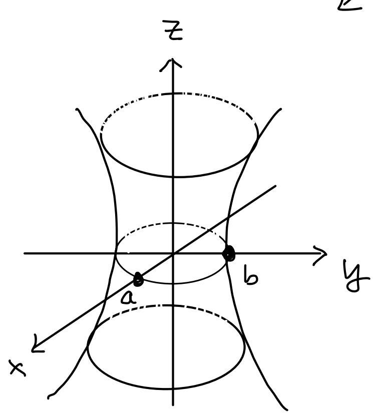


$r^2-z^2=1$

same for each direction of θ $\rightsquigarrow$



Hyperboloid of 1 sheet



scaling back to

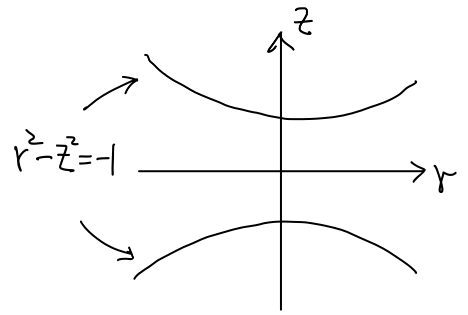
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

eg3 $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$

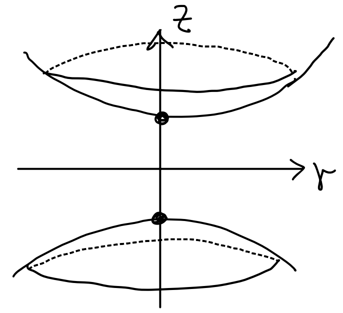
Similarly, after scaling, looks like

$$x^2+y^2-z^2=-1 \iff r^2-z^2=-1 \text{ for } r^2=x^2+y^2$$

$z^2-r^2=1$ with polar coord. on xy-plane



same for each direction of θ $\rightsquigarrow$



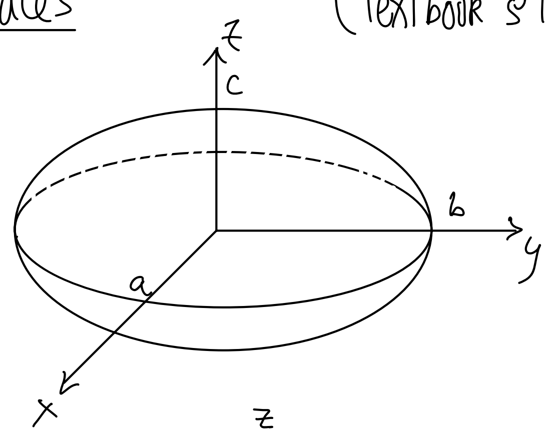
Hyperboloid of 2 sheets

In summary, we have

Graphs of Standard Quadratic Surfaces (Textbook §12.6)

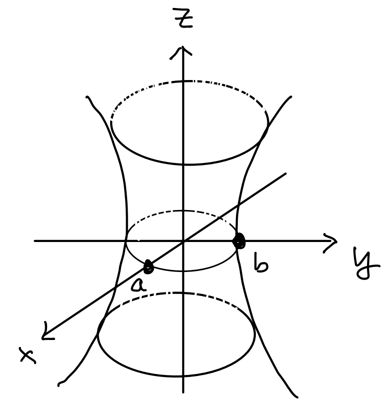
Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



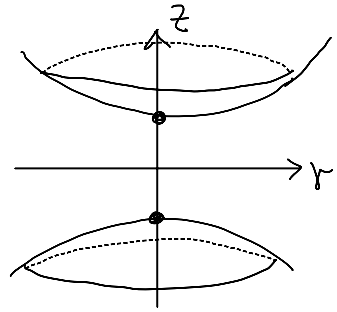
Hyperboloid of 1 sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



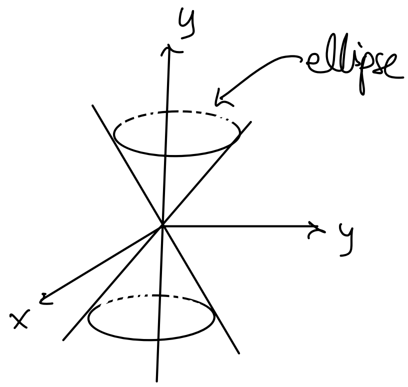
Hyperboloid of 2 sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$



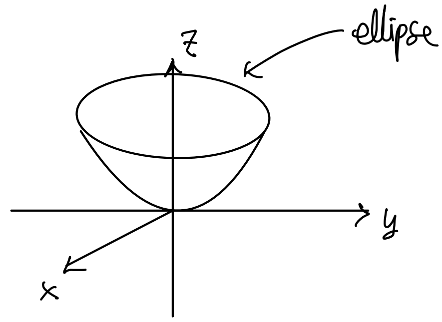
Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$



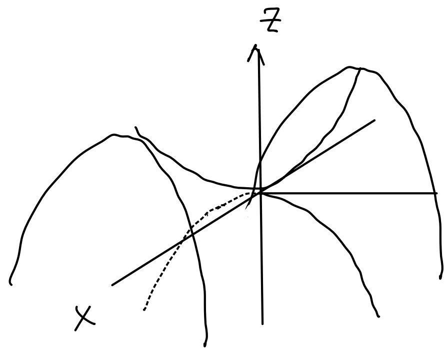
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



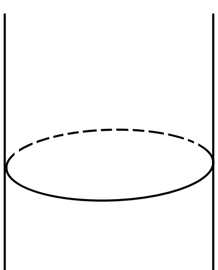
Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

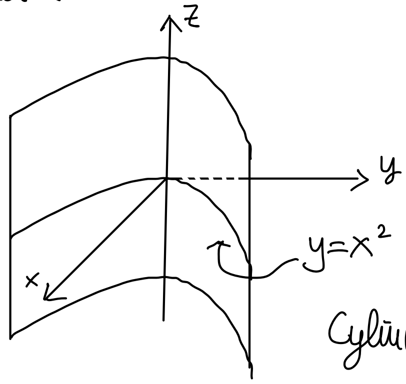


Degenerate to cylinders over conic sections

eg: Equation involve NO z variables



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cylinder of ellipse



and etc.

$y = x^2$

Cylinder of parabola

Other degenerate cases

eg $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=0$ & $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=-1$

Fact Any quadratic constraint $g(x,y,z)=c$ can be transformed to one of the standard forms by a change of coordinates.

Remark: As in 2-variables, only ellipsoid is closed and bounded

Further examples

eg1 Find the point on the ellipse

$$x^2+xy+y^2=9 \qquad \text{(Check: it is really a ellipse!)}$$

with maximum x-coordinate.

Soln: Let $f(x,y)=x$

$g(x,y)=x^2+xy+y^2$

Maximize f under constraint $g=9$.

Consider $F(x,y,\lambda)=x-\lambda(x^2+xy+y^2-9)$

$$\left\{\begin{aligned} 0&=\frac{\partial F}{\partial x}=1-\lambda(2x+y)\\ 0&=\frac{\partial F}{\partial y}=-\lambda(x+2y)\\ 0&=\frac{\partial F}{\partial \lambda}=-(x^2+xy+y^2-9)\end{aligned}\right. \iff \left\{\begin{aligned} &\lambda(2x+y)=1 &&-(1)\\ &\lambda(x+2y)=0 &&-(2)\\ &x^2+xy+y^2=9 &&-(3)\end{aligned}\right.$$

$(1) \Rightarrow \lambda \neq 0$

then $(2) \Rightarrow \quad x+2y=0 \quad \Rightarrow \quad x=-2y$

sub into $(3) \Rightarrow \quad (-2y)^2+(-2y)y+y^2=9$

$\Rightarrow \quad y=\pm\sqrt{3}$

Hence $(x,y)=(-2\sqrt{3},\sqrt{3}), \quad (2\sqrt{3},-\sqrt{3})$ are the critical points

Comparing the values $2\sqrt{3}>-2\sqrt{3}$

$\Rightarrow$ maximum value of x-coordinate is $2\sqrt{3}$.

(at the point $(2\sqrt{3},-\sqrt{3})$) #

eg2 Find the point(s) on the hyperboloid

$xy-yz-zx=3$ (Check: it is really a hyperboloid ↑ of 2 sheets)

closest to the origin

Soln let $f(x,y,z)=x^2+y^2+z^2$

$g(x,y,z)=xy-yz-zx$

Minimize f under constraint $g=3$

Consider $F(x,y,z,\lambda)=x^2+y^2+z^2-\lambda(xy-yz-zx-3)$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2x - \lambda(y-z) & \text{—— (1)} \\ 0 = \dfrac{\partial F}{\partial y} = 2y - \lambda(x-z) & \text{—— (2)} \\ 0 = \dfrac{\partial F}{\partial z} = 2z - \lambda(-y-x) & \text{—— (3)} \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(xy - yz - zx - 3) & \text{—— (4)} \end{cases}$$

If $\lambda = 0$, then (1), (2) & (3) $\Rightarrow$ $x = y = z = 0$ contradicting (4).

$\therefore$ $\lambda \neq 0$. Then (1), (2) & (3) $\Rightarrow$
$$\begin{cases} y - z = \frac{2}{\lambda}x & \text{—— (5)} \\ x - z = \frac{2}{\lambda}y & \text{—— (6)} \\ x + y = -\frac{2}{\lambda}z & \text{—— (7)} \end{cases}$$

(5) − (6) $\Rightarrow$ $y - x = \frac{2}{\lambda}(x-y) \Rightarrow (1+\frac{2}{\lambda})(x-y) = 0$ —— (8)

(7) − (6) $\Rightarrow$ $y + z = -\frac{2}{\lambda}(z+y) \Rightarrow (1+\frac{2}{\lambda})(y+z) = 0$ —— (9)

If $1 + \frac{2}{\lambda} = 0$, i.e. $\lambda = -2$, then (5), (6) & (7) gives $x + y - z = 0$

$\Rightarrow$ $0 = (x+y-z)^2 = x^2 + y^2 + z^2 + 2xy - 2yz - 2xz = x^2 + y^2 + z^2 + 6$

which is a contradiction $\therefore$ $1 + \frac{2}{\lambda} \neq 0$

(8) & (9) then $\Rightarrow$ $x = y = -z$

Sub into (4) $\Rightarrow$ $3x^2 = 3 \Rightarrow x = \pm 1$

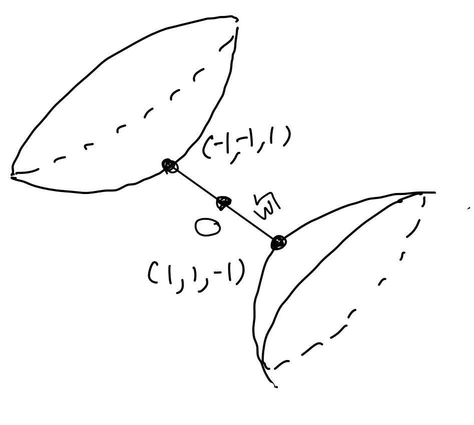
$\therefore$ $(x, y, z) = \pm(1, 1, -1)$ (& $\lambda = 1$)

$f(1,1,-1) = f(-1,-1,1) = 3$

$\Rightarrow$ closest points are $\pm(1,1,-1)$

with corresponding distance $= \sqrt{3}$

#



Lagrange Multipliers with multiple Constraints

Let
- $f, g_1, \cdots, g_k : \Omega \to \mathbb{R}$ be C^1 functions, ($\Omega \subseteq \mathbb{R}^n$, open)
- $S = \{\vec{x} \in \Omega : g_i(\vec{x}) = c_i \text{ for } i = 1, \cdots, k\}$

Suppose
- $\vec{a}$ is a local extremum of f on S
- $\vec{\nabla} g_1(\vec{a}), \cdots, \vec{\nabla} g_k(\vec{a})$ are linearly independent vectors

Then

$$\begin{cases} \vec{\nabla} f(\vec{a}) = \sum_{i=1}^{k} \lambda_i \vec{\nabla} g_i(\vec{a}) \\ g_i(\vec{a}) = c_i, \quad i = 1, \cdots, k \end{cases}$$

for some Lagrange multipliers $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$.

Same as 1 constraint,

Finding extrema of $f(\vec{x})$ with constraints $g_i(\vec{x}) = c_i$, $i = 1, \cdots, k$

$$\Updownarrow$$

Finding extrema of $F(\vec{x}, \lambda_1, \cdots, \lambda_k) = f(\vec{x}) - \sum_{i=1}^{k} \lambda_i (g_i(\vec{x}) - c_i)$ without constraint

(but more variables: adding λ_i as new variables) eg1 Maximize $f(x,y,z) = x^2 + 2y - z^2$

on the line $L: \begin{cases} 2x - y = 0 \\ y + z = 0 \end{cases}$ in $\mathbb{R}^3$

(Given that maximum exists)

Soln Let $g_1(x,y,z) = 2x - y$

$g_2(x,y,z) = y + z$

Maximize f subject to constraints $\begin{cases} g_1 = 0 \\ g_2 = 0 \end{cases}$

$\left[\begin{array}{l} f \text{ is 2-degree poly,} \\ g_1, g_2 \text{ are degree 1 polynomials} \end{array} \Rightarrow f, g_1, g_2 \text{ are } C^1 \right]$

$\left. \begin{array}{l} \vec{\nabla} g_1 = (2 \quad -1 \quad 0) \\ \vec{\nabla} g_2 = (0 \quad 1 \quad 1) \end{array} \right\}$ are linearly independent (prove it!)

Consider

$$F(x,y,z,\lambda_1,\lambda_2) = x^2 + 2y - z^2 - \lambda_1(2x - y) - \lambda_2(y + z)$$

$$\left\{ \begin{array}{l} 0 = \frac{\partial F}{\partial x} = 2x - 2\lambda_1 \\ 0 = \frac{\partial F}{\partial y} = 2 + \lambda_1 - \lambda_2 \\ 0 = \frac{\partial F}{\partial z} = -2z - \lambda_2 \\ 0 = \frac{\partial F}{\partial \lambda_1} = -(2x - y) \\ 0 = \frac{\partial F}{\partial \lambda_2} = -(y + z) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} x = \lambda_1 & \text{——— (1)} \\ \lambda_2 = \lambda_1 + 2 & \text{——— (2)} \\ \lambda_2 = -2z & \text{——— (3)} \\ 2x = y & \text{——— (4)} \\ y = -z & \text{——— (5)} \end{array} \right.$$

(1) & (3) sub into (2)

$$-2z = x+2 \quad \text{———} \quad (6)$$

(4) & (5) $\Rightarrow$ $\quad 2x = y = -z \quad$ ——— (7)

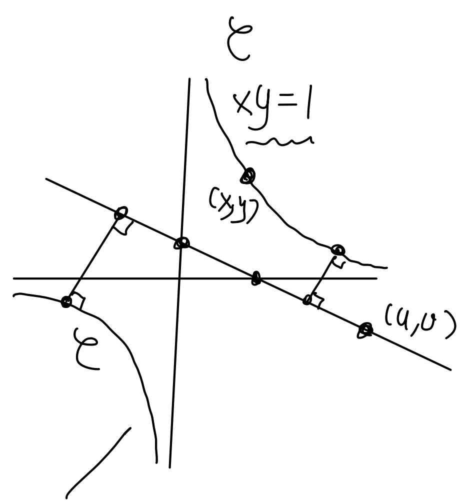
sub into (6) $\quad 4x = x+2 \Rightarrow x = \frac{2}{3}$

sub into (7) $\Rightarrow \quad y = \frac{4}{3}, \; z = -\frac{4}{3}$

$\Rightarrow$ max occurs at $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$

with value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{3}\right) - \left(\frac{4}{3}\right)^2$

(check!) $= \frac{4}{3}$ #



eg2 Find the distance between the hyperbola $\mathcal{E}: xy=1$ and the line $L: x+4y = \frac{15}{8}$

Soln: Let

$$f(x,y,u,v) = (x-u)^2 + (y-v)^2$$

Minimize f under constraints

$$g_1(x,y,u,v) = xy = 1$$

$$g_2(x,y,u,v) = u+4v = \frac{15}{8}$$

$$\vec{\nabla} g_1 = [\,y \quad x \quad 0 \quad 0\,]$$
$$\vec{\nabla} g_2 = [\,0 \quad 0 \quad 1 \quad 4\,]$$

$\vec{\nabla} g_1$ & $\vec{\nabla} g_2$ are linearly independent

$\Leftrightarrow \quad (x, y) \neq (0,0)$ (Can you prove it?)

Consider

$$F(x,y,u,v,\lambda_1,\lambda_2) = (x-u)^2+(y-v)^2-\lambda_1(xy-1)-\lambda_2\left(u+4v-\frac{15}{8}\right)$$

$$\begin{cases} 0=\dfrac{\partial F}{\partial x} = 2(x-u)-\lambda_1 y & \text{———— (1)} \\ 0=\dfrac{\partial F}{\partial y} = 2(y-v)-\lambda_1 x & \text{———— (2)} \\ 0=\dfrac{\partial F}{\partial u} = -2(x-u)-\lambda_2 & \text{———— (3)} \\ 0=\dfrac{\partial F}{\partial v} = -2(y-v)-4\lambda_2 & \text{———— (4)} \\ 0=\dfrac{\partial F}{\partial \lambda_1} = -(xy-1) & \text{———— (5)} \\ 0=\dfrac{\partial F}{\partial \lambda_2} = -\left(u+4v-\frac{15}{8}\right) & \text{———— (6)} \end{cases}$$

Case 1 If $\lambda_1=0$ or $\lambda_2=0$, then

$x=u$ & $y=v$

sub into (6) $\Rightarrow \quad x=\frac{15}{8}-4y$

sub into (5) $\Rightarrow \quad \left(\frac{15}{8}-4y\right)y=1$

$4y^2-\frac{15}{8}y+1=0$ has no (real) solution

Case 2 $\lambda_1 \neq 0$ & $\lambda_2 \neq 0$.

Then (3) & (4) $\Rightarrow$

$$\left.\begin{array}{l} \dfrac{x-u}{y-v} = \dfrac{1}{4} \\ \\ \text{\& } (1) \& (2) \Rightarrow \\ \\ \dfrac{x-u}{y-v} = \dfrac{y}{x} \end{array}\right\} \Rightarrow x = 4y$$

sub. into (5) $\quad (4y)y = 1 \Rightarrow y = \pm\frac{1}{2}$

$\therefore (x,y) = \pm(2, \frac{1}{2}) \quad (\neq (0,0))$

Then for $(2,\frac{1}{2})$, $\dfrac{2-u}{\frac{1}{2}-v} = \dfrac{1}{4} \Rightarrow$

$$\left.\begin{array}{l} 4u - v = \dfrac{15}{2} \\ u + 4v = \dfrac{15}{8} \end{array}\right\}$$

together (6)

$\Rightarrow (u,v) = (\frac{15}{8}, 0)$

Similarly for $(-2, -\frac{1}{2})$, we have $(u,v) = (-\frac{225}{136}, \frac{15}{17})$ (Ex!)

Comparing the values $\quad f(2, \frac{1}{2}, \frac{15}{8}, 0) = \frac{17}{64} \left(= (\text{dist})^2\right)$ (check!)

$f(-2, -\frac{1}{2}, -\frac{225}{136}, \frac{15}{17}) = \dots > \frac{17}{64}$

check

$\Rightarrow$ distance between $\mathcal{C}$ and $L = \frac{\sqrt{17}}{8}$ (Check) ※