Lagrange Multipliers

(A method for *finding extrema under constraints*)

eg1 In previous example of finding global max/min of $f(x,y) = x^2+2y^2-x+3$ for $x^2+y^2 \le 1$, one need to find (in step 2) the max/min values of f on the boundary $x^2+y^2=1$.

In otherwords, finding global max/min

of $$f(x,y) = x^2+2y^2-x+3$$

under constraint $$g(x,y) = x^2+y^2 = 1$$

Another typical example :

eg2 Find the point on the parabola $x^2=4y$ closest to $(1,2)$.

i.e. Find (global) minimum of

$$f(x,y) = (x-1)^2+(y-2)^2$$

(equivalent to, but easier than $\sqrt{(x-1)^2+(y-2)^2}$)

under constraint

$$g(x,y) = x^2-4y = 0 .$$

Remark : In both examples, constraints are expressed as level set. $g=c$ for some constant c.

Thm (<u>Lagrange Multipliers</u>)

Let
- $f, g : \Omega \to \mathbb{R}$ be <u>C^1</u> functions, ($\Omega \subset \mathbb{R}^n$ open)
- $S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$ be a level set of g

Suppose
- $\vec{a} \in S$ is a local <u>extremum of f restricted to S</u>
  (i.e. under the constraint $g = c$)
- $\vec{\nabla} g(\vec{a}) \neq \vec{0}$

Then
- $\vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a})$ for some $\lambda \in \mathbb{R}$
- $g(\vec{a}) = c$

where λ is called a <u>Lagrange Multiplier</u>

<u>Reduction to unconstrainted problem</u> (By Lagrange Multiplier)

Finding extrema of $f(\vec{x})$ with constraint $g(\vec{x}) = c$

$$\Updownarrow$$

Finding extrema of $F(\vec{x}, \lambda) = f(\vec{x}) - \lambda(g(\vec{x}) - c)$
without constraint

(but more variables: adding λ as a new variable)

<u>Idea</u>: $F(\vec{x}, \lambda) = F(x_1, \cdots, x_n, \lambda)$ is $n+1$ variables

$$\vec{0} = \vec{\nabla} F = \left(\frac{\partial F}{\partial x_1}, \cdots \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial \lambda} \right)$$

$\vec{0} \in \mathbb{R}^{n+1}$; $\vec{\nabla}$: $n+1$-variable

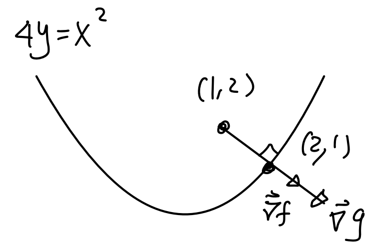
$$\begin{cases} 0 = \dfrac{\partial F}{\partial x_i} = \dfrac{\partial}{\partial x_i}\left(f - \lambda(g-c)\right) = \dfrac{\partial f}{\partial x_i} - \lambda \dfrac{\partial g}{\partial x_i} & \forall\, i = 1, \cdots, n \\ 0 = \dfrac{\partial F}{\partial \lambda} = \dfrac{\partial}{\partial \lambda}\left(f - \lambda(g-c)\right) = -(g-c) \end{cases}$$

$$\Leftrightarrow \begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = c \end{cases}$$

$\vec{\nabla}$: n-variable $\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$

<u>eg2</u> (cont'd) minimize $f(x,y) = (x-1)^2 + (y-2)^2$

under constraint $g(x,y) = x^2 - 4y = 0$



<u>Soln</u>: Consider

$$F(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - 0)$$
$$= (x-1)^2 + (y-2)^2 - \lambda(x^2 - 4y)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2(x-1) - \underline{2\lambda x} & \text{—— (1)} \\ 0 = \dfrac{\partial F}{\partial y} = 2(y-2) + 4\lambda & \text{—— (2)} \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(x^2 - 4y) & \text{—— (3)} \end{cases}$$

$(2) \Rightarrow \quad 2\lambda = 2 - y$

put in (1) $\Rightarrow \quad 2(x-1) - (2-y)x = 0$

$\Rightarrow \quad y = \frac{2}{x}$

put in (3) $\Rightarrow \quad x^2 - \frac{8}{x} = 0 \quad \Rightarrow \quad x = 2$

Hence $\quad y = 1$

$\therefore$ $(2,1)$ is the only critical.

$\Rightarrow$ f has a minimum at $(2,1) \in g^{-1}(0)$

with value $f(2,1) = (2-1)^2 + (1-2)^2 = 2$

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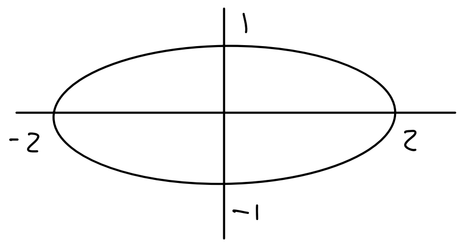
eg2 Maximize $f(x,y) = xy^2$ on the ellipse $x^2 + 4y^2 = 4$

Soln : $\begin{cases} f(x,y) = xy^2 \\ g(x,y) = x^2 + 4y^2 \end{cases}$



& $F(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - 4)$

$= xy^2 - \lambda(x^2 + 4y^2 - 4)$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = y^2 - 2\lambda x \\ 0 = \frac{\partial F}{\partial y} = 2xy - 8\lambda y \\ 0 = \frac{\partial F}{\partial \lambda} = -(x^2 + 4y^2 - 4) \end{cases}$$

By simple calculation, we have

$$(x,y) = (\pm 2, 0) \text{ or } \left(\pm\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}\right)$$

$$\left(= (2,0), (-2,0), \left(\frac{2}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) \left(\frac{2}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right) \left(\frac{-2}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) \left(\frac{-2}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right) \right)$$

Comparing values of f at these 6 critical points:

$$f(\pm 2, 0) = 0$$

$$f\left(\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{2}{\sqrt{3}} \cdot \frac{2}{3} = \frac{4}{3\sqrt{3}} \longleftarrow \text{max}$$

$$f\left(-\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\frac{2}{\sqrt{3}} \cdot \frac{2}{3} = -\frac{4}{3\sqrt{3}} \longleftarrow \text{min}$$

$\therefore$ For $f(x,y)$ on $g(x,y)=4$, the

global max value $= \dfrac{4}{3\sqrt{3}}$ at $\left(\dfrac{2}{\sqrt{3}}, \pm\sqrt{\dfrac{2}{3}}\right)$

global min value $= -\dfrac{4}{3\sqrt{3}}$ at $\left(\dfrac{-2}{\sqrt{3}}, \pm\sqrt{\dfrac{2}{3}}\right)$

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eg 1 (cont'd) Using Lagrange multiplier, find global max/min of

$$f(x,y) = x^2+2y^2-x+3 \quad \text{on} \quad x^2+y^2=1$$

(Step 2 of the original global max/min problem on $x^2+y^2 \le 1$)

Soln: Let $f(x,y) = x^2+2y^2-x+3$

$g(x,y) = x^2+y^2-1$

& $F(x,y,\lambda) = x^2+2y^2-x+3-\lambda(x^2+y^2-1)$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2x-1-2\lambda x \\ 0 = \dfrac{\partial F}{\partial y} = 4y-2\lambda y \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(x^2+y^2-1) \end{cases}$$

Simple calculation $\Rightarrow$

$(x,y) = (\pm 1, 0)$, $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

Comparing values:

$f\left(-\frac{1}{2}, \pm\sqrt{\frac{3}{2}}\right) = \frac{21}{4}$ $\leftarrow$ max (on $x^2+y^2=1$)

$f(1,0) = 3$ $\leftarrow$ min (on $x^2+y^2=1$)

$f(-1,0) = 5$

$\therefore$ global max of f $\underline{\text{on } x^2+y^2=1}$ $= \frac{21}{4}$ at $\left(-\frac{1}{2}, \pm\sqrt{\frac{3}{2}}\right)$

global min of f $\underline{\text{on } x^2+y^2=1}$ $= 3$ at $(1, 0)$

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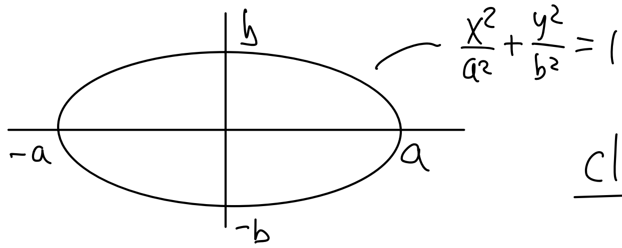
Classification of Quadratic Constraints

2-variables : $g(x,y) = Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F$

(Conic Section)

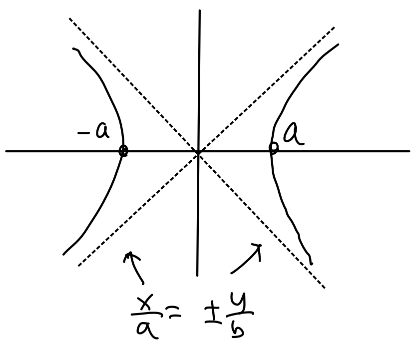
Typical examples for level curve $g(x,y) = c$

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a, b > 0$ Ellipse (circle if $a=b$)



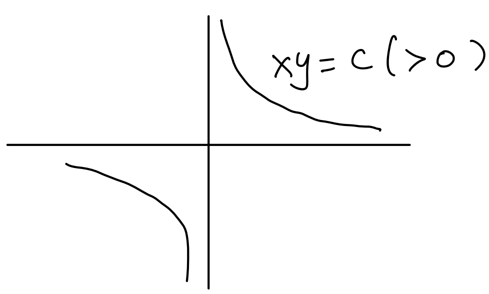
closed & bounded

(ii) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $a, b > 0$ Hyperbola



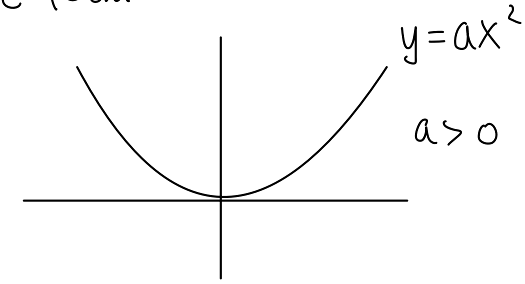
unbounded

($xy = c$, $c \neq 0$, is also a hyperbola)



(iii) $y = ax^2, \ a \neq 0$ <u>Parabola</u>

(only 1 quadratic term)



<u>unbounded</u>

(iv) <u>Degenerate Cases</u> $(a>0, b>0)$

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \longrightarrow$ a point $(0,0)$
- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1 \longrightarrow$ empty set
- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0 \longrightarrow \dfrac{x}{a} = \pm \dfrac{y}{b}$ a pair of intersecting lines

  $(xy = 0)$
- $x^2 = c \longrightarrow x = \pm\sqrt{c}$
  - a pair of parallel line if $c>0$
  - a "double" line if $c=0$
  - empty set if $c<0$

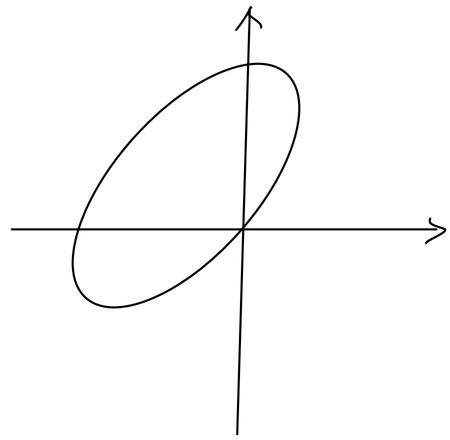
<u>Fact</u>: By a change of coordinates, any <u>quadratic constraint</u> $g(x,y)=c$ can be transformed to one of the form above

Hence level sets of quadratic constraints are ellipse, hyperbola, parabola, & degenerated cases

eg $17x^2 - 12xy + 8y^2 + 16\sqrt{5}x - 8\sqrt{5}y = 0$

$$\iff \frac{u^2}{1^2} + \frac{v^2}{2^2} = 1$$

where $\begin{cases} u = \dfrac{2x-y}{\sqrt{5}} + 1, \\ v = \dfrac{x+2y}{\sqrt{5}} \end{cases}$



Remark: Ellipse is closed and bounded $\Rightarrow$ Any continuous $f(x,y)$ restricted to an ellipse has global max & min.

Not the case for hyperbola & parabola.

Quadratic Constraint for 3-variables

$$g(x,y,z) = Ax^2 + By^2 + Cz^2 + 2Pxy + 2Qyz + 2Rzx + Dx + Ey + Fz + G$$

Some typical examples of $g = \text{const}$.

eg 1 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Ellipsoid