eg 1 $\quad f(x,y) = x^2+y^2$

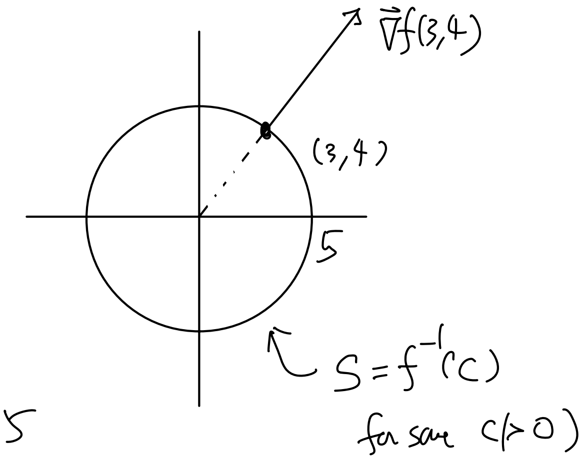
$\vec{\nabla} f = (2x, 2y)$

For instance, let $c=25$

then $(3,4) \in f^{-1}(25)$

$\vec{\nabla} f(3,4) = (6,8) \perp S = f^{-1}(25)$

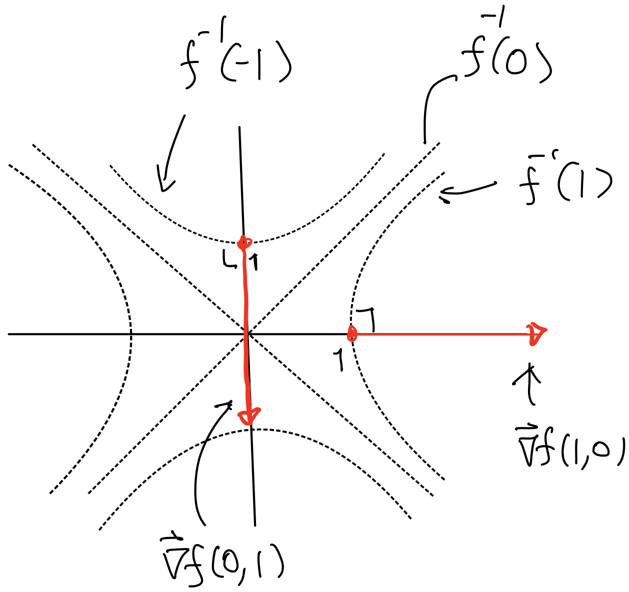
at the point $(3,4)$



eg 2 $\quad f(x,y) = x^2-y^2$

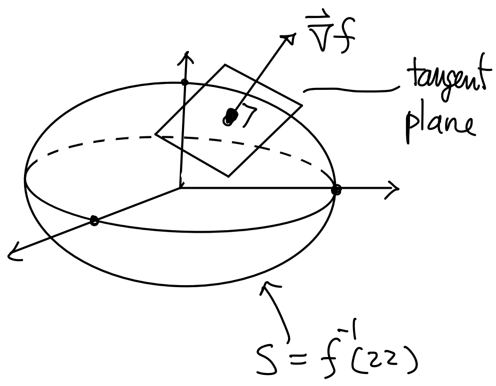
$\vec{\nabla} f(x,y) = (2x, -2y)$

(Ex!) Try other points



eg 3 $\quad S: x^2+4y^2+9z^2 = 22$ (Ellipsoid)

Find equation of tangent plane of S at the point $(3,1,1)$



(Check: $(3,1,1)$ is on S)

Soln: $\quad$ Let $f(x,y,z) = x^2+4y^2+9z^2$

Then $S = f^{-1}(22)$

$\vec{\nabla} f = (2x, 8y, 18z)$

$\vec{\nabla} f(3,1,1) = (6, 8, 18) \perp S$ at $(3,1,1)$

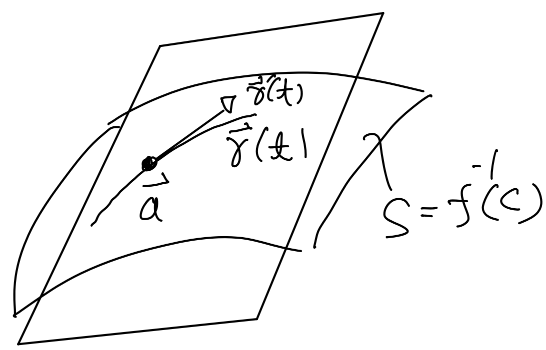
$\therefore$ $\vec{\nabla} f(3,1,1)$ is a normal to the tangent plane at the point $(3,1,1)$.

$$\Rightarrow \quad \vec{\nabla} f(3,1,1) \cdot (x-3, y-1, z-1) = 0$$

is a equation of the tangent plane.

$$\therefore \quad 6(x-3) + 8(y-1) + 18(z-1) = 0$$

or $\quad 3x + 4y + 9z = 22$ is the required equation of the tangent at $(3,1,1)$.



<u>Proof of the Thm:</u> $(\vec{\nabla} f \perp S)$

Let $\vec{\gamma}(t)$ be a curve on S passing through the point $\vec{a}$ such that $\vec{\gamma}(0) = \vec{a}$

Then $\quad f(\vec{\gamma}(t)) = c, \ \forall t$

Chain rule $\Rightarrow \quad 0 = \left.\frac{d}{dt}\right|_{t=0} f(\vec{\gamma}(t)) = \left.\vec{\nabla} f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)\right|_{t=0}$

tangent vector of the curve at $\gamma(t)$ which is also a tangent vector of S

i.e. $\quad \vec{\nabla} f(\vec{a}) \cdot \vec{\gamma}'(0) = 0$

$\vec{\nabla} f(\vec{a}) \perp$ all curves on S at $\vec{a}$

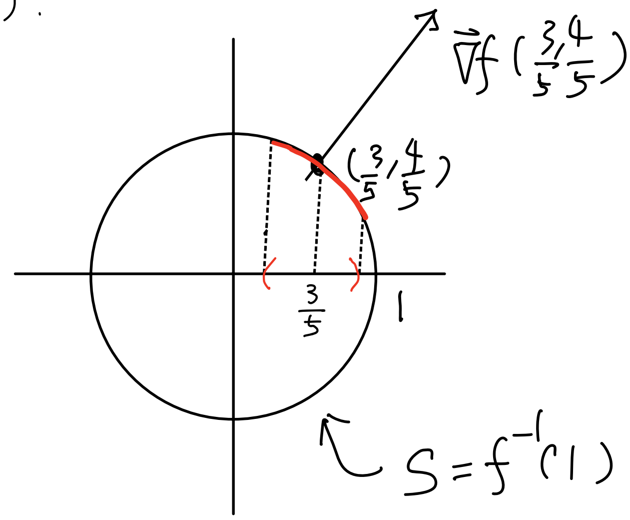
$\Rightarrow \quad \vec{\nabla} f(\vec{a}) \perp S$ at $\vec{a}$. $\#$

Another Application of Chain Rule:

Implicit Differentiation

eg 1 $\quad C: x^2+y^2=1$ $\quad$ (y can be solved in term of x (for most x))

Find $\dfrac{dy}{dx}$ at $\left(\frac{3}{5}, \frac{4}{5}\right)$.



Soln: Near the point $\left(\frac{3}{5}, \frac{4}{5}\right)$

$$x^2+(y(x))^2=1$$

$$\Rightarrow \quad \frac{d}{dx}\left(x^2+(y(x))^2\right)=0$$

$$2x+2y\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{x}{y} \qquad (\text{provided } y\neq 0)$$

At the point $\left(\frac{3}{5}, \frac{4}{5}\right) \Rightarrow \dfrac{dy}{dx}=-\dfrac{3}{4}$.

Remark: One cannot solve y as a function of x near the points $(1,0)$ and $(-1,0)$ which correspond to "$y=0$".

eg 2 Consider $S: x^3+z^2+ye^{xz}+z\cos y=0$

Given that z can be regarded as a function $z=z(x,y)$ of (independent) variables x, y locally near the point $(0,0,0)$. (Clearly $(0,0,0)\in S$)

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(0,0,0)$

Soln:
$$\frac{\partial}{\partial x}\left(x^3+z^2+ye^{xz}+z\cos y\right)=0$$

$$3x^2+2z\frac{\partial z}{\partial x}+y\,e^{xz}\frac{\partial}{\partial x}(xz)+\frac{\partial z}{\partial x}\cos y=0$$

$$3x^2+2z\frac{\partial z}{\partial x}+ye^{xz}\left(z+x\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial x}\cos y=0$$

$$(3x^2+yze^{xz})+(2z+xye^{xz}+\cos y)\frac{\partial z}{\partial x}=0$$

$$\Rightarrow\quad \frac{\partial z}{\partial x}=-\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y}$$

(provided $2z+xye^{xz}+\cos y\neq 0$)

$$\Rightarrow\quad \frac{\partial z}{\partial x}(0,0)=0 \qquad \left(\text{since } 2\cdot 0+0\cdot 0e^{0\cdot 0}+\cos 0=1\neq 0\right)$$

Similarly, $\frac{\partial}{\partial y}\left(x^3+z^2+ye^{xz}+z\cos y\right)=0$

$$\Rightarrow\quad 2z\frac{\partial z}{\partial y}+e^{xz}+ye^{xz}\left(x\frac{\partial z}{\partial y}\right)+\frac{\partial z}{\partial y}\cos y-z\sin y=0$$

(check) $\Rightarrow$
$$\frac{\partial z}{\partial y}=\frac{z\sin y-e^{xz}}{2z+xye^{xz}+\cos y}$$
(provided $2z+xye^{xz}+\cos y\neq 0$)

$\Rightarrow \quad \frac{\partial z}{\partial y}(0,0) = -1$.

※

Finding Extrema (Maximum & Minimum)

Def: Let $\begin{cases} \bullet\ f : A \to \mathbb{R}, \quad A \subset \mathbb{R}^n \text{ (may not open)} \\ \bullet\ \vec{a} \in A \end{cases}$

(1) f is said to have a global (absolute) maximum at $\vec{a}$

if $\quad f(\vec{a}) \geq f(\vec{x}) \quad \forall \vec{x} \in A$

(2) f is said to have a local (relative) maximum at $\vec{a}$

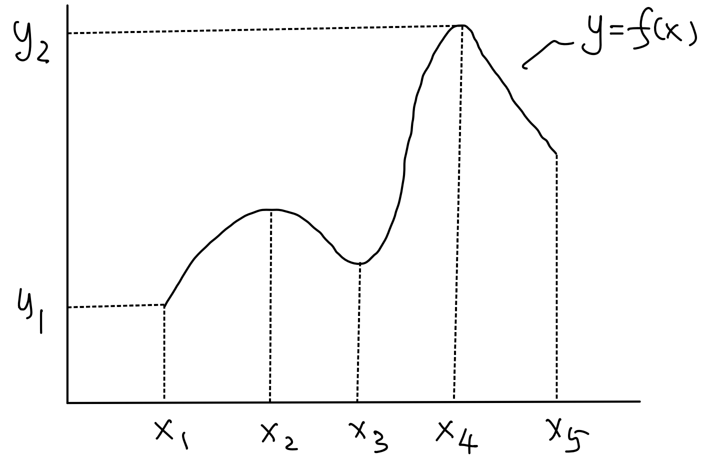
if $\quad f(\vec{a}) \geq f(\vec{x}) \quad \forall \vec{x} \in A$ "near" $\vec{a}$

(i.e. $\exists\, \varepsilon > 0$ s.t. $f(\vec{a}) \geq f(\vec{x}),\ \forall \vec{x} \in A \cap B_\varepsilon(\vec{a})$)

(3) Similar definitions for global (absolute) minimum and local (relative) minimum by changing the inequality to $f(\vec{a}) \leq f(\vec{x})$.

Remark: Global Extremum (max/min) is also a local extremum.

eg1 $f : [x_1, x_5] \longrightarrow \mathbb{R}$



Global: max : x_4; min : x_1

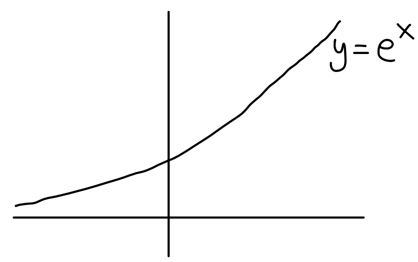
Local: max : x_2 & x_4; min : x_1, x_3 & x_5

max value : y_2

min value : y_1

eg 2 (NOT every function has global max/min)

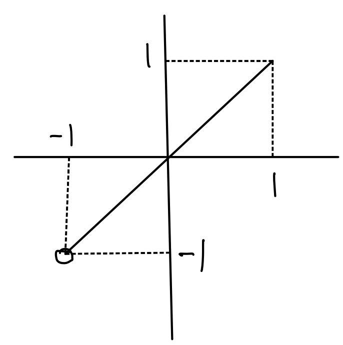
(i) $f(x) = e^x$ on $\mathbb{R}$ (Domain unbounded)



No global min : $\lim_{x \to -\infty} f(x) = 0$ & $f(x) > 0 \ \forall x \in \mathbb{R}$

No global max : $\lim_{x \to +\infty} f(x) = +\infty$

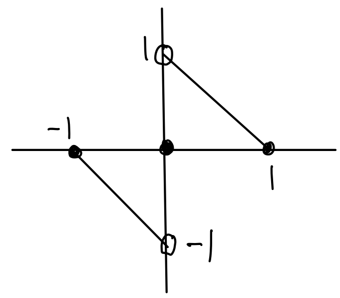
(ii) $f(x) = x$ on $(-1, 1]$ (Domain not closed)



Has global max at $x = 1$
(with max. value $= f(1) = 1$)

but no global min

(iii) $f = \begin{cases} 1-x, & 0<x\le 1 \\ 0, & x=0 \\ -1-x, & -1\le x<0 \end{cases}$ (discontinuous)



No global max and global min

(since f is discontinuous)

Extreme Value Thm (EVT)

Let
- $A \subseteq \mathbb{R}^n$ be closed and bounded,
- $f: A \to \mathbb{R}$ be continuous

Then f has global max and min.

Remarks: (1) "Compact" = closed and bounded

(2) The Thm is a sufficient, but not a necessary condition.

Def: Let
- $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ (not necessary open)
- $\vec{a} \in \operatorname{Int}(A)$

Then $\vec{a}$ is called a critical point of f if

either (1) $\vec{\nabla} f(\vec{a})$ DNE (does not exist)

or (2) $\vec{\nabla} f(\vec{a}) = \vec{0}$

(i.e. either "$\frac{\partial f}{\partial x_i}(\vec{a})$ DNE for some $i=1,\cdots,n$" or "$\frac{\partial f}{\partial x_i}(\vec{a})=0$ for all $i=1,\cdots,n$")

Thm (First Derivative Test)

Suppose $f : A \to \mathbb{R}$ ($A \subset \mathbb{R}^n$) attains a local extremum at $\vec{a} \in \mathrm{Int}(A)$, then $\vec{a}$ is a critical point of f.

Pf: Suppose f has a local extremum at $\vec{a} \in \mathrm{Int}(A)$

If $\vec{\nabla} f(\vec{a})$ DNE, then $\vec{a}$ is a critical point.

If $\vec{\nabla} f(\vec{a})$ exists, then

all $\frac{\partial f}{\partial x_i}(\vec{a})$ exists

$\forall i$, Consider 1-variable function $g_i(t) = f(\vec{a} + t\vec{e_i})$

$$\vec{e_i} = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow \text{ith component} \\ \\ \\ \end{array} \end{array}$$

Then $g_i(0) = f(\vec{a}) \underset{\leq}{\overset{\geq}{\text{or}}} f(\vec{a} + t\vec{e_i})$ for $|t|$ small.

$$\Rightarrow 0 = g_i'(0) = \left.\frac{d}{dt}\right|_{t=0} f(\vec{a} + t\vec{e_i}) = \frac{\partial f}{\partial x_i}(\vec{a}) \quad (\forall i)$$

$\therefore \quad \vec{\nabla} f(\vec{a}) = \vec{0}$ $\quad$ ※

Strategy for finding Extrema

$f: A \to \mathbb{R}$

(1) Find critical points of f in $\text{Int}(A)$.

(2) Study f on boundary ∂A:

find max/min of f on ∂A.

(3) Compare values of f at points found in steps (1) & (2).

---

eg 1 Find global max/min of

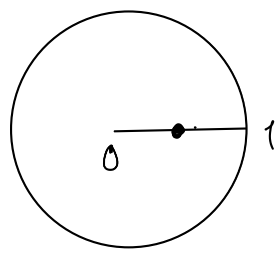
cts

closed & bdd

$f(x,y) = x^2 + 2y^2 - x + 3$ for $x^2+y^2 \le 1$ $\left(A = \{x^2+y^2 \le 1\} \right)$

Soln: **Step 1** Critical points in $\text{Int}\{x^2+y^2\le 1\}$

$$\text{Int}\{x^2+y^2\le 1\} = \{x^2+y^2<1\}$$



(f is a polynomail, $\vec{\nabla} f$ always exist)

$$\vec{\nabla} f = (2x-1, 4y)$$

$$\vec{\nabla} f = \vec{0} \iff \begin{cases} 2x-1=0 \\ 4y=0 \end{cases} \iff (x,y) = \left(\tfrac{1}{2}, 0\right)$$

(the only critical pt. in $\text{Int}(A)$)

Clearly $(\frac{1}{2}, 0) \in \mathrm{Int}(A)$

& $\quad f(\frac{1}{2}, 0) = (\frac{1}{2})^2 + 0 - \frac{1}{2} + 3 = \frac{11}{4}$

<u>Step 2</u> Study f on $\partial\{x^2+y^2 \leq 1\} = \{x^2+y^2=1\}$

parametrize the boundary $\{x^2+y^2=1\}$ by angle θ

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$$

Then on $\partial\{x^2+y^2 \leq 1\}$,

$$f(\theta) = f(\cos\theta, \sin\theta) = \cos^2\theta + 2\sin^2\theta - \cos\theta + 3$$

$$= -\cos^2\theta - \cos\theta + 5$$

$$= -(\cos\theta + \tfrac{1}{2})^2 + \tfrac{21}{4}$$

max. value of f on $\partial A = \frac{21}{4}$ (at $\cos\theta = -\frac{1}{2} \Rightarrow (x,y) = (-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$)

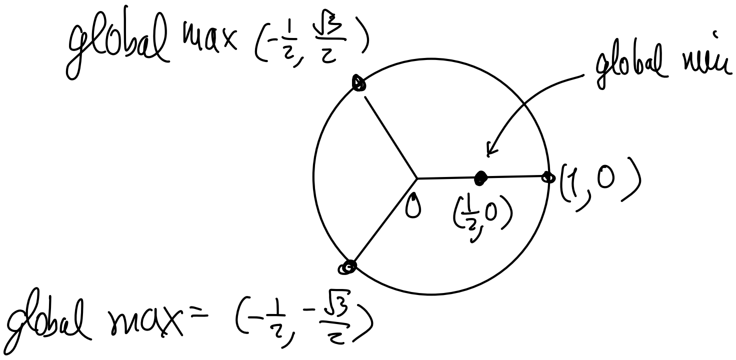
min. value of f on $\partial A = -(1+\frac{1}{2})^2 + \frac{21}{4} = 3$
(at $\cos\theta = 1 \Rightarrow (x,y) = (1,0)$)

<u>Step 3</u> Compare values of f at points from steps 1 & 2.

$f(\frac{1}{2}, 0) = \frac{11}{4}$ value of critical pt. in $\mathrm{Int}(A)$

$f(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{21}{4}$ max value of f on ∂A

$f(1,0) = 3$ min value of f on ∂A



$\Rightarrow$ max value $= \frac{21}{4}$ at the max points $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$
(global)

& min value $= \frac{11}{4}$ at the min point $(\frac{1}{2}, 0)$
(global)

$\#$