((Cont'd) (b) (of eg 1):

(In a sense $(1.1, 1.9) \approx (1, 2)$)

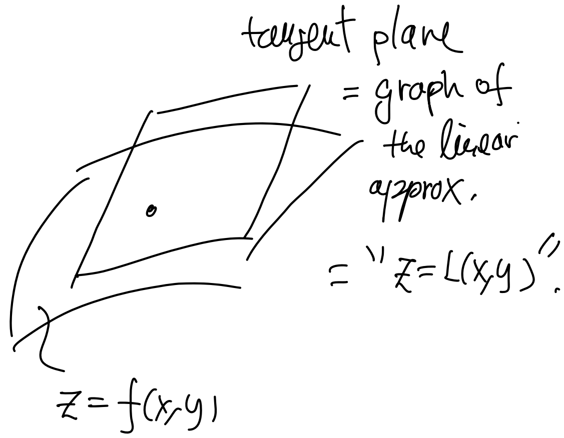
$$f(1.1, 1.9) \approx L(1.1, 1.9)$$

$$= 2 + 4(1.1-1) + (1.9-2)$$

$$= 2 + 4 \times 0.1 + (-0.1)$$

$$= 2.3$$

(c) The equation of the tangent plane of $z = f(x,y)$ at the point $(x,y) = (1,2)$ is



$$z = L(x,y)$$

$$= 2 + 4(x-1) + (y-2)$$

i.e. $z = 4x + y - 4$

$$(\ 4x + y - z = 4\ )$$

\#

<u>eg 2</u> Is $f(x,y) = \sqrt{|xy|}$ differentiable at $(0,0)$?

<u>Soln</u>:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \cdots = 0 \qquad \text{(Similarly! Ex!)}$$

Linearization $L(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0)$

$$= 0 + 0\cdot x + 0\cdot y$$

$$\equiv 0 \qquad \text{(is the zero function)}$$

Error term $\varepsilon(x,y) = f(x,y) - L(x,y)$

$$= f(x,y) = \sqrt{|xy|}\,.$$

$$\lim_{(x,y)\to(0,0)} \frac{|\varepsilon(x,y)|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}$$

$$= \lim_{r\to 0} \frac{\sqrt{r^2|\cos\theta\sin\theta|}}{r} = \lim_{r\to 0} \sqrt{|\cos\theta\sin\theta|}$$

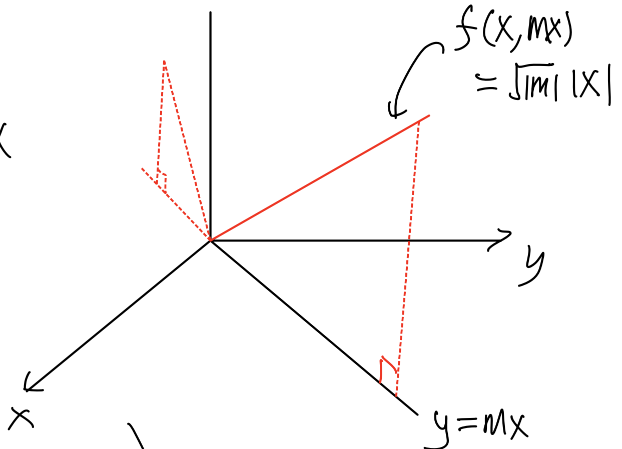
↑ different directions (different θ) gives different limits.

$\therefore$ limit DNE !

$\therefore$ $f = \sqrt{|xy|}$ is not differentiable at $(0,0)$.

Remark: In this example, along the straight line $y = mx$

$$f(x,y) = \sqrt{|xmx|} = \sqrt{|m|}\,|x|$$



(only "approximated" by $L(x,y)$ in the $m=0$ situation.)

(Note: "Differentiability" $\Rightarrow$ we can approximate all (infinity many) directions by information along x & y direction. ($x_1, \dots, x_n$) in general)

Thm If $f(\vec{x})$ is differentiable at $\vec{a}$, then $f(\vec{x})$ is continuous at $\vec{a}$.

Pf: $f(\vec{x}) = L(\vec{x}) + \mathcal{E}(\vec{x})$ is differentiable $\Leftrightarrow \lim_{\vec{x}\to\vec{a}} \frac{|\mathcal{E}(\vec{x})|}{\|\vec{x}-\vec{a}\|} = 0$

$$= f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \mathcal{E}(\vec{x})$$

$$\Rightarrow |f(\vec{x}) - f(\vec{a})| \le \left| \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) \right| + |\mathcal{E}(\vec{x})| \quad \text{(Triangle ineq.)}$$

$$\text{(Cauchy-Schwarz)} \le \left(\sqrt{\left(\frac{\partial f}{\partial x_i}(\vec{a})\right)^2} + \frac{|\mathcal{E}(\vec{x})|}{\|\vec{x}-\vec{a}\|} \right) \cdot \|\vec{x}-\vec{a}\|$$

$\to 0$ by Squeeze Thm & Differentiability #

Thm If $f, g : \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open) are differentiable at $\vec{a} \in \Omega$,

then (1) $f(\vec{x}) \pm g(\vec{x})$, $cf(\vec{x})$, $f(\vec{x})g(\vec{x})$ are differentiable at $\vec{a}$.

(2) $\frac{f(\vec{x})}{g(\vec{x})}$ is differentiable at $\vec{a}$ if $g(\vec{a}) \neq 0$

(3) (Special case of Chain Rule)
For 1-variable function $h(x)$ differentiable at $f(\vec{a})$, $h \circ f$ is differentiable at $\vec{a}$.

A sufficient condition for differentiability:

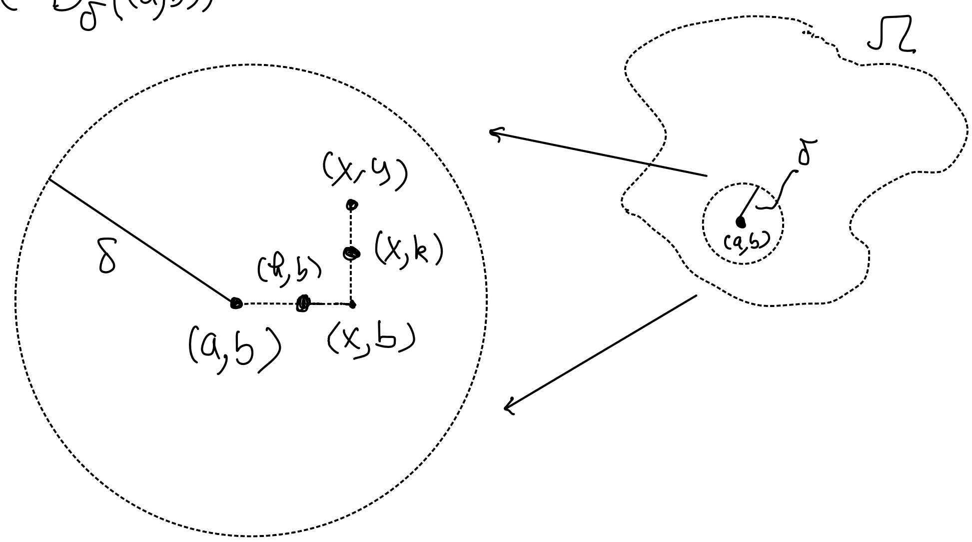
Thm Let $\Omega \subseteq \mathbb{R}^n$ be open, f be C^1 on Ω, then f is differentiable on Ω

(The assumption requires all $\frac{\partial f}{\partial x_i}$ exist on Ω, not just at a single pt. $\vec{a}$)

Pf: (We prove it for 2-variables, similar proof for general case)

Suppose $(a,b) \in \Omega$ & let $B_\delta((a,b)) \subseteq \Omega$.

For any $(x,y) \in B_\delta((a,b))$



$$f(x,y) - f(a,b) = \underbrace{f(x,y) - f(x,b)} + \underbrace{f(x,b) - f(a,b)}$$

(by Mean Value Thm) $= f_y(x,k)(y-b) + f_x(h,b)(x-a)$

where k between y & b and h between x & a

$$\frac{|\varepsilon(x,y)|}{\|(x,y)-(a,b)\|} = \frac{|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{\|(x,y)-(a,b)\|}$$

$$= \frac{|f_y(x,k)(y-b) + f_x(h,b)(x-a) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{\|(x,y)-(a,b)\|}$$

$$= \frac{|[f_x(h,b)-f_x(a,b)](x-a) + [f_y(x,k)-f_y(a,b)](y-b)|}{\|(x,y)-(a,b)\|}$$

$$\leq \frac{\sqrt{(f_x(h,b)-f_x(a,b))^2 + (f_y(x,k)-f_y(a,b))^2}\ \sqrt{(x-a)^2+(y-b)^2}}{\|(x,y)-(a,b)\|}$$

$$= \sqrt{(f_x(h,b)-f_x(a,b))^2 + (f_y(x,k)-f_y(a,b))^2} \longrightarrow 0$$

as $(x,y)\to(a,b)$

since f_x & f_y are continuous $\left(\begin{matrix} x\to a \Rightarrow h\to a \\ y\to b \Rightarrow k\to b \end{matrix}\right)$

$\therefore$ f is differentiable at $(a,b)\in\Omega$.

Since (a,b) is arbitrary, f is differentiable on Ω. $\#$

<u>egs</u>: (1) constant functions $f(\vec{x}) = c$ are differentiable

(2) coordinate functions $f(\vec{x}) = x_i$ are differentiable

(3) (1) & (2) $\Rightarrow$

$$f(\vec{x}) = a + b_1x_1 + \cdots + b_nx_n \quad \text{is differentiable}$$

(Question: What is the linearization $L(\vec{x})$ at $(0,\cdots,0)$?)

(4) Polynomials & rational functions are differentiable (in their domain of definition).

(5) If $f(\vec{x})$ is differentiable, then

$e^{f(\vec{x})}$, $\sin(f(\vec{x}))$, $\cos(f(\vec{x}))$ are differentiable.

And
$$\left.\begin{array}{ll} \ln(f(\vec{x})) & \text{when } f(\vec{x})>0 \\ \sqrt{f(\vec{x})} & \text{when } f(\vec{x})>0 \\ |f(\vec{x})| & \text{when } f(\vec{x})\neq 0 \\ \ln|f(\vec{x})| & \text{when } f(\vec{x})\neq 0 \end{array}\right\} \text{are differentiable}$$

in particular (eg:) $\dfrac{e^{\sqrt{4+\sin(x^2+xy)}}}{\ln(1+\cos|x^2y|)}$ is differentiable in the domain of definition

eg: $f(x,y,z) = xe^{x+y} - \ln(x+z)$ $\left(= xe^{x+y} - \log(x+z)\right)$

Domain of $f = \{(x,y,z)\in\mathbb{R}^3 : x+z>0\}$ is open

$$\frac{\partial f}{\partial x} = e^{x+y} + xe^{x+y} - \frac{1}{x+z}$$

$$\frac{\partial f}{\partial y} = xe^{x+y}$$

$$\frac{\partial f}{\partial z} = -\frac{1}{x+z}$$

$x+z>0$

all terms are continuous in the domain of f.

$\Rightarrow$ f is C^1 (on its domain)

$\Rightarrow$ f is differentiable (on its domain).

Gradient and Directional Derivative

Def: Let $\begin{cases} \bullet\ f : \Omega \to \mathbb{R}, & (\Omega \subseteq \mathbb{R}^n, \text{ open}) \\ \bullet\ \vec{a} \in \Omega \end{cases}$

Then the gradient vector of f at $\vec{a}$ is defined to be

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

Remark: Using $\vec{\nabla} f$, linearization of f at $\vec{a}$ can be written as

$$\boxed{\begin{aligned} L(\vec{x}) &= f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) \\ &= f(\vec{a}) + \vec{\nabla} f(\vec{a}) \bullet (\vec{x} - \vec{a}) \end{aligned}}$$

eg: $f(x,y) = x^2 + 2xy$

$\frac{\partial f}{\partial x} = 2x + 2y$, $\quad \frac{\partial f}{\partial y} = 2x$

$\therefore \quad \vec{\nabla} f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x + 2y, 2x)$

(eg. $\vec{\nabla} f(1,2) = (6,2)$)

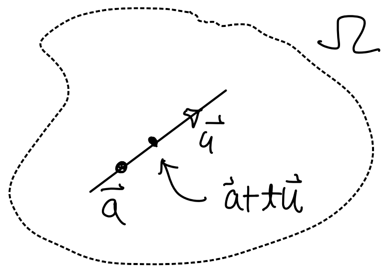
Def: Let
- $f: \Omega \to \mathbb{R}$, ($\Omega \subseteq \mathbb{R}^n$, open)
- $\vec{a} \in \Omega$
- $\vec{u} \in \mathbb{R}^n$ be a unit vector, i.e. $\|\vec{u}\| = 1$.

Then the directional derivative of f in the direction of $\vec{u}$ at $\vec{a}$ is defined to be

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

(= rate of change of f in the direction of $\vec{u}$ at the point $\vec{a}$)

Remark: If $\vec{u} = (0, \cdots, 1, \cdots, 0) = \vec{e}_j$, $\leftarrow$ jth component, $j = 1, \cdots, n$



$$D_{\vec{e}_j} f(\vec{a}) = \frac{\partial f}{\partial x_j}(\vec{a})$$

Thm Suppose f is differentiable at $\vec{a}$.

Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$, then

$$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$$

eg: Let $f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$.

Find the rate of change of f at $(1, \sqrt{2})$ in the direction of $\vec{v} = (1, -1)$ (not necessary unit).

Remark: $\vec{v} \neq \vec{0} \in \mathbb{R}^n$, not necessary unit, then

the direction of $\vec{v}$ is $\frac{\vec{v}}{\|\vec{v}\|}$ (a unit vector).

Soln: Let $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{1^2+(-1)^2}}(1,-1) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\sin^{-1}\left(\frac{x}{y}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}\left(\sin^{-1}\left(\frac{x}{y}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(-\frac{x}{y^2}\right)$$

Note $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous near $(1, \sqrt{2})$

$\Rightarrow$ f is C^1 near $(1, \sqrt{2})$

$\Rightarrow$ f is differentiable at $(1, \sqrt{2})$

$$\Rightarrow \quad D_{\vec{u}} f(1, \sqrt{2}) = \vec{\nabla} f(1, \sqrt{2}) \cdot \vec{u}$$

$$= \left(\frac{\partial f}{\partial x}(1, \sqrt{2}), \frac{\partial f}{\partial y}(1, \sqrt{2})\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

$$= \left(1, -\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \qquad \text{(Check!)}$$

$$= \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{2}. \quad \#$$

Pf: (Differentiable $\Rightarrow$ $D_{\vec{u}}f(\vec{a}) = \vec{\nabla}f(\vec{a})\cdot\vec{u}$)

Let $L(\vec{x})$ be linearization of $f(\vec{x})$ at $\vec{a}$.

& $f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$

$$= f(\vec{a}) + \vec{\nabla}f(\vec{a})\cdot(\vec{x}-\vec{a}) + \varepsilon(\vec{x})$$

with $\dfrac{|\varepsilon(\vec{x})|}{\|\vec{x}-\vec{a}\|} \to 0$ as $\vec{x}\to\vec{a}$.

Put $\vec{x} = \vec{a}+t\vec{u}$, we have

$$f(\vec{a}+t\vec{u}) - f(\vec{a}) = \vec{\nabla}f(\vec{a})\cdot(t\vec{u}) + \varepsilon(\vec{a}+t\vec{u})$$

$$= t\,(\vec{\nabla}f(\vec{a})\cdot\vec{u}) + \varepsilon(\vec{a}+t\vec{u})$$

$$\frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t} = \vec{\nabla}f(\vec{a})\cdot\vec{u} + \frac{\varepsilon(\vec{a}+t\vec{u})}{t}$$

$$\left|\frac{\varepsilon(\vec{a}+t\vec{u})}{t}\right| = \frac{|\varepsilon(\vec{a}+t\vec{u})|}{\|(\vec{a}+t\vec{u})-\vec{a}\|} \to 0 \quad \text{as } \vec{a}+t\vec{u}\to\vec{a}$$

(ie. $t\to 0$)

$$\therefore\ D_{\vec{u}}f(\vec{a}) = \lim_{t\to 0}\frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t} = \vec{\nabla}f(\vec{a})\cdot\vec{u}$$

※

Geometric Meanings of Gradient $\vec{\nabla} f$

> At a point $\vec{a}$, f increases (decreases) most rapidly in the direction of $\vec{\nabla} f(\vec{a})$ $(-\vec{\nabla} f(\vec{a}))$ at a rate of $\|\vec{\nabla} f(\vec{a})\|$

Idea: If f is differentiable at $\vec{a}$, then

$$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u} \quad (\text{for } \|\vec{u}\| = 1)$$

Cauchy-Schwarz $\Rightarrow$

$$|\vec{\nabla} f(\vec{a}) \cdot \vec{u}| \leq \|\vec{\nabla} f(\vec{a})\| \|\vec{u}\| = \|\vec{\nabla} f(\vec{a})\|$$

i.e. $$-\|\vec{\nabla} f(\vec{a})\| \leq \vec{\nabla} f(\vec{a}) \cdot \vec{u} \leq \|\vec{\nabla} f(\vec{a})\|$$

"=" holds $\Leftrightarrow$ $\vec{u} = \dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$

"=" holds $\Leftrightarrow$ $\vec{u} = -\dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$

#