eg of higher order derivatives (order ≥ 3)

Previous eg : $f(x,y) = x\sin y + y^2 e^{2x}$

$\Rightarrow \quad f_{xy} = \cos y + 4ye^{2x} = f_{yx}$

$\Rightarrow$ (3rd order) $f_{xyx} = (f_{xy})_x = 8ye^{2x} = (f_{yx})_x = f_{yxx}$

$f_{xyy} = (f_{xy})_y = -\sin y + 4e^{2x} = (f_{yx})_y = f_{yxy}$

$\vdots$ (Ex : other 3rd order partial derivatives)

One can calculate similarly up to any order

Question Is it always true that $f_{xy} = f_{yx}$?

Answer : No.

Counterexample :

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

By definition,

$$f_{xy}(0,0) = (f_x)_y(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$\therefore$ we need to calculate the 1st order partial derivative $f_x(0,h)$ and $f_x(0,0)$.

For $(0,h)$, $\quad f_x(x,y) = \dfrac{\partial}{\partial x}\left(\dfrac{xy(x^2-y^2)}{x^2+y^2} \right)$

$$= \frac{(x^2+y^2)(3x^2y-y^3) - xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$$

$$\Rightarrow f_x(0,h) = \frac{-h^5}{h^4} = -h$$

For $(0,0)$, $f_x(0,0) = \lim_{k\to 0} \frac{f(k,0)-f(0,0)}{k}$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$\therefore \quad f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{-h-0}{h} = -1 .$$

Similarly, $f_{yx}(0,0) = 1$ (Ex!)

(An easy way to see this make sense: $f(x,y) = -f(y,x)$)

Hence $f_{xy}(0,0) \neq f_{yx}(0,0)$!

<u>Question</u>: When do we have $f_{xy} = f_{yx}$?

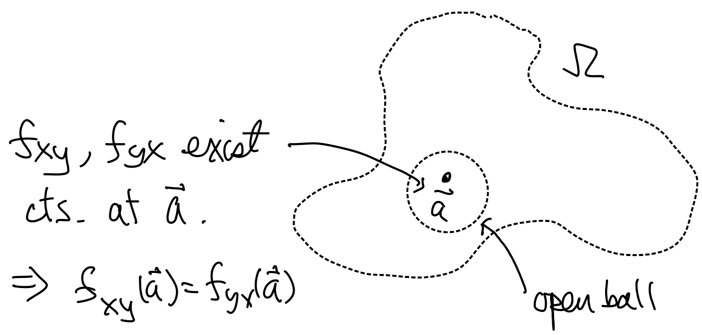
> <u>Thm</u> (<u>Clairaut's Thm</u> / <u>Mixed Derivatives Thm</u>)
>
> Let $f: \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open)
>
> If f_{xy} & f_{yx} <u>exist</u> and are <u>continuous</u> on Ω, then
>
> $$f_{xy} = f_{yx} \text{ on } \Omega.$$

Actually, one can prove a stronger version:

Thm Let
- $f : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open)
- $\vec{a} \in \Omega$

If
- f_{xy} & f_{yx} exist in an open ball containing $\vec{a}$, and
- f_{xy} & f_{yx} are continuous at $\vec{a}$,

then $f_{xy}(\vec{a}) = f_{yx}(\vec{a})$.



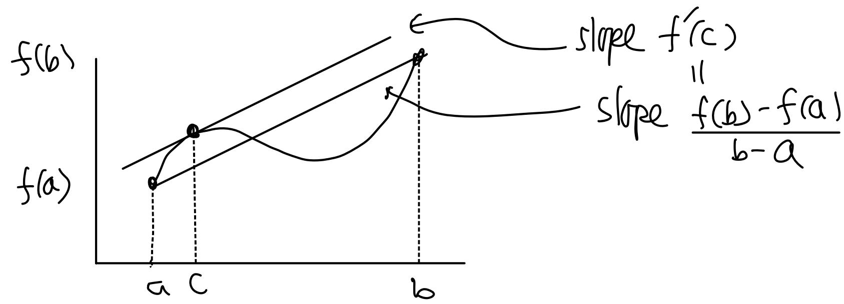
Recall

Mean Value Theorem for 1-variable Function

Let $f : [a,b] \to \mathbb{R}$,
- continuous on $[a,b]$ &
- differentiable on (a,b)

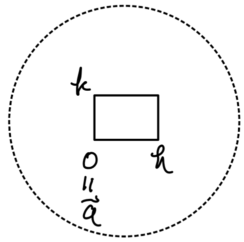
Then $\exists\, c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$



Pf of Clairaut's Thm

We may assume $\vec{a} = (0,0) \in \Omega$



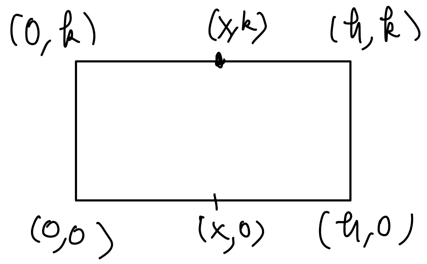
and need to show $f_{xy}(0,0) = f_{yx}(0,0)$.

Let $h, k > 0$ and $[0,h] \times [0,k] \subset \Omega$

$$-g(0)$$

Define $\alpha = f(h,k) - f(0,k) - f(h,0) + f(0,0)$

$$g(h)$$



Let $g(x) = f(x,k) - f(x,0)$, $0 \le x \le h$

Then $\alpha = g(h) - g(0)$

$$g'(x) = f_x(x,k) - f_x(x,0)$$

Mean Value Thm $\Rightarrow$ $\exists\, h_1 \in (0,h)$ such that

$$\frac{g(h) - g(0)}{h} = g'(h_1)$$

i.e. $$\frac{\alpha}{h} = f_x(h_1,k) - f_x(h_1,0)$$

$$\alpha = h\left[f_x(h_1, k) - f_x(h_1,0)\right]$$

Mean Value Thm again $\Rightarrow$ $\exists\, k_1 \in (0,k)$ such that

$$\frac{f_x(h_1,k) - f_x(h_1,0)}{k} = (f_x)_y(h_1,k_1)$$

$\therefore \quad \alpha = hk \, f_{xy}(h_1, k_1)$

Similarly, $\exists (h_2, k_2) \in (0,h) \times (0,k)$ such that (Ex!)

$$\alpha = hk \, f_{yx}(h_2, k_2)$$ (By interchanging the role of x & y)

$\therefore \quad f_{xy}(h_1, k_1) = f_{yx}(h_2, k_2)$

Letting $h, k \to 0^+ \Rightarrow h_1, k_1 \to 0$ & $h_2, k_2 \to 0$

$\therefore \quad f_{xy}(0,0) = f_{yx}(0,0)$ since f_{xy} & f_{yx} are continuous at $\vec{a} = (0,0)$.

$\divideontimes$

Def Let $f : \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^n$, open)

Then
- f is called a <u>C^k function</u> if <u>all partial derivatives</u> of f up to <u>order k</u> <u>exist</u> and are <u>continuous</u> on Ω
- f is called a <u>C^∞ function</u> if f is C^k for all $k \geq 0$.

---

egs: (1) If f is continuous (0-order partial derivative) then f is C^0.

(2) If f is C^2, then $f, f_x, f_y, f_{xx}, f_{xy} = f_{yx}, f_{yy}$ exist & are all continuous. (by Clairaut's)

(3) Polynomials, Rational functions, exponential, logarithm, trigonometric functions are C^∞ function on their domains of definition.

& hence their sum / difference / product / quotient / compositions are C^∞ function on their domains of definition.

explicit eg: $e^{x^2-y} \sin\left(\frac{x}{y}\right)$ (except $y=0$)

on domain of definition $= \mathbb{R}^2 \setminus \{x\text{-axis}\}$

Generalization of Clairaut's Thm

> If f is C^k on an open set $\Omega \subseteq \mathbb{R}^n$, then the order of (taking) differentiation does not matter for all partial derivatives up to order k.

eg If $f(x,y,z)$ is C^3, then

$$f_{xz} = f_{zx}, \quad f_{xyz} = f_{xzy} = f_{zxy} = f_{zyx}$$

$$\vdots \qquad\qquad = f_{yzx} = f_{yxz}$$

etc.

$$f_{xxy} = f_{xyx} = f_{yxx} \text{ and etc.}$$

(Mid term up to here, generalization of Clairaut's Thm)

Differentiability

Recall : 1-variable : f is differentiable at a

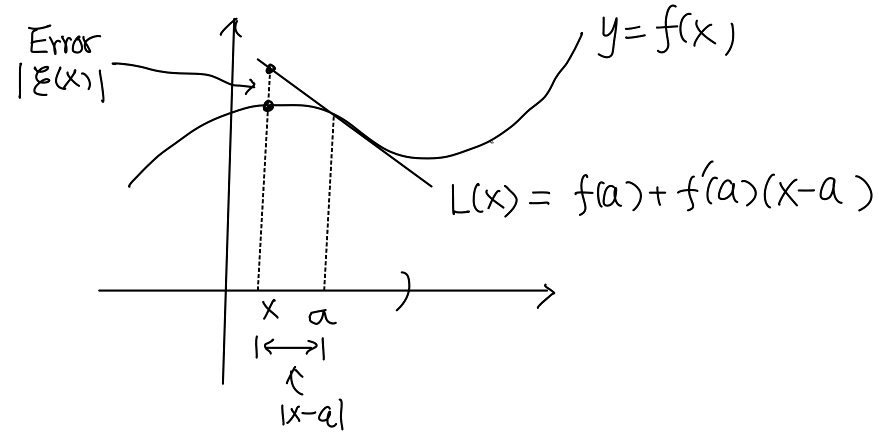
if $\quad f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists

which is equivalent to

Linear Approximation of f at the point a :

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}$$

$L(x)$ is the "best" linear function (deg ≤ 1, poly) to approximate $f(x)$ near a



What does it mean by "best" ?

$$\lim_{x \to 0} \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right|$$

$$\parallel$$

Answer : $\displaystyle \lim_{x \to 0} \frac{|f(x) - L(x)|}{|x - a|} = 0 \qquad \left(\lim_{x \to a} \frac{|\mathcal{E}(x)|}{|x-a|} = 0 \right)$

where $f(x) - L(x)$ is usually referred as the "error" term $\mathcal{E}(x) = f(x) - L(x)$.

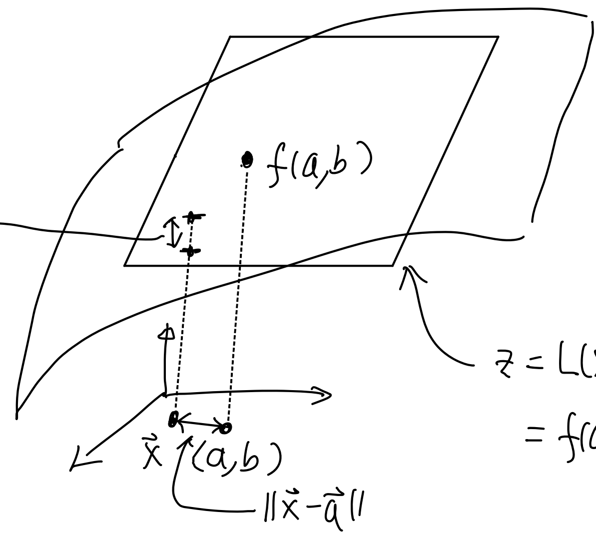
Higher dimension's analog:

linear function (deg ≤ 1, poly)

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

and want

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$



Def: Let
- $f : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$, open
- $\vec{a} = (a_1, \cdots, a_n) \in \Omega$

Then f is said to be <u>differentiable at $\vec{a}$</u>

if (1) $\frac{\partial f}{\partial x_i}(\vec{a})$ exists for all $i = 1, \cdots, n$

(2) In the linear approximation for $f(\vec{x})$ at $\vec{a}$

$$f(\vec{x}) = \underbrace{f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i)}_{L(\vec{x}) \text{ linear approx.}} + \varepsilon(\vec{x})$$

error term

the <u>error term</u> $\varepsilon(\vec{x})$ satisfies

$$\lim_{\vec{x} \to \vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x} - \vec{a}\|} = 0.$$

(A differentiable function is one which can be well approximated by a linear function locally.)

Remark: $L(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x_i}(\vec{a})}_{\substack{\uparrow \\ \text{slope of } f \text{ in} \\ x_i\text{-direction at } \vec{a}}} \underbrace{(x_i - a_i)}_{\Delta x_i}$

- $L(\vec{x})$ is a deg ≤ 1 polynomial
- $L(\vec{a}) = f(\vec{a})$
- $\frac{\partial L}{\partial x_i}(\vec{a}) = \frac{\partial f}{\partial x_i}(\vec{a})$ (Easy Ex!)
- The graph of $y = L(\vec{x})$ is a n-plane tangent to the graph of $y = f(\vec{x})$ (which is a surface) at the point $\vec{x} = \vec{a}$.

eg 1: $f(x,y) = x^2 y$

(1) Show that f is differentiable at $(1,2)$

(2) Approximate $f(1.1, 1.9)$ using linearization.

(3) Find tangent plane of $z = f(x,y)$ at $(1,2,\underbrace{2}_{f(1,2)})$.

Soln: (1) $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = x^2$

$\frac{\partial f}{\partial x}(1,2) = 4$, $\frac{\partial f}{\partial y}(1,2) = 1$

$\therefore$ The linearization at $(1,2)$ is

$$L(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$$

$$= 2 + 4(x-1) + (y-2) \quad (= 4x+y-2)$$

with error term

$$\varepsilon(x,y) = f(x,y) - L(x,y)$$

$$= x^2y - [2+4(x-1)+(y-2)]$$

$$\lim_{(x,y)\to(1,2)} \frac{|\varepsilon(x,y)|}{\|(x,y)-(1,2)\|}$$

$$= \lim_{(x,y)\to(1,2)} \frac{|x^2y - 2 - 4(x-1) - (y-2)|}{\sqrt{(x-1)^2+(y-2)^2}} \qquad \left(\text{let } \begin{matrix} h = x-1 \\ k = y-2 \end{matrix}\right)$$

$$= \lim_{(h,k)\to(0,0)} \frac{|(1+h)^2(k+2) - 2 - 4h - k|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|h^2k + 2hk + 2h^2|}{\sqrt{h^2+k^2}} \qquad \left(\text{let } \begin{cases} h = r\cos\theta \\ k = r\sin\theta \end{cases}\right)$$

$$= \lim_{r\to 0} \frac{|r^3\cos^2\theta\sin\theta + 2r^2\cos\theta\sin\theta + 2r^2\cos^2\theta|}{r}$$

$$= \lim_{r\to 0} r\,|r\cos^2\theta\sin\theta + 2\cos\theta\sin\theta + 2\cos^2\theta|$$

$= 0$. by Squeeze Thm

$\therefore$ f is differentiable at $(1,2)$. (To be cont'd)