Finding Limit using Polar Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \quad "(x,y)\to(0,0) \Leftrightarrow r\to 0"$$

egs (1) Find limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ using polar coordinates.

Soln :

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to 0} r\left(\cos^3\theta + \sin^3\theta\right) \qquad (r\geq 0)$$

$$\left| r(\cos^3\theta+\sin^3\theta)\right| \leq r\left(|\cos\theta|^3+|\sin\theta|^3\right) \leq 2r$$

Since $\displaystyle\lim_{r\to 0} 2r = 0$, squeeze thm $\Rightarrow$

$$\lim_{r\to 0} r\left(\cos^3\theta+\sin^3\theta\right) = 0$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$$

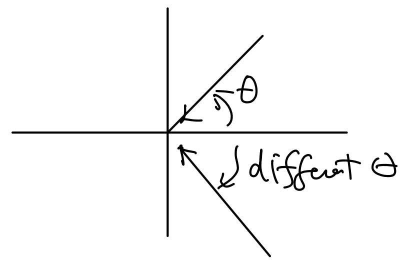
(2) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)}$ DNE.

Soln:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)} = \lim_{r\to 0} \frac{\cancel{r^2}\left(\cos^2\theta+\cos\theta\sin\theta\right)}{2\cancel{r^2}}$$

$$= \frac{\cos^2\theta+\cos\theta\sin\theta}{2}$$

Different θ means approaching $(0,0)$ in different directions



$\therefore$ Different directions $\Rightarrow$ different θ

$\Rightarrow$ different limits.

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+xy}{2(x^2+y^2)}$ DNE.

(3) $\lim\limits_{(x,y)\to(0,0)} xy\ln(x^2+y^2)$

$$= \lim_{r\to 0} r^2\cos\theta\sin\theta\ln r^2 = \underbrace{\left(\lim_{r\to 0} r^2\ln r\right)}_{=0}\cdot 2\cos\theta\sin\theta$$

$= 0$

$\Bigg(\left| r^2\cos\theta\sin\theta\ln r^2 \right| \le 2r^2\ln\frac{1}{r}$, for r near 0

L'Hopital's Rule $\Rightarrow \lim\limits_{r\to 0} r^2\ln\frac{1}{r} = 0$ (review) $\Bigg)$

Iterated Limit

(1) $\lim\limits_{x\to a} \lim\limits_{y\to b} f(x,y) \overset{\text{def}}{=\!=} \lim\limits_{x\to a}\left(\lim\limits_{y\to b} f(x,y)\right)$

i.e. 1^{st} take limit as $y\to b$, then take limit as $x\to a$.

(2) Similarly for $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y)$

(3) Are they equal ? $\left(\lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y) = \lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y)\ ?\right)$

(4) Relation to $\lim\limits_{(x,y)\to(a,b)} f(x,y)$

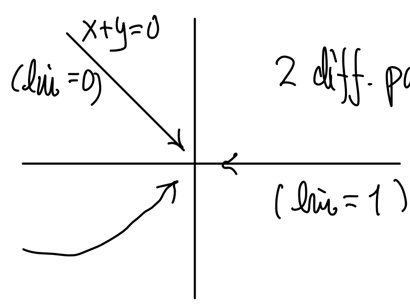
eg: Consider $f(x,y) = \dfrac{x+y}{x-y}$

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x+y}{x-y} = \lim_{x\to 0}\frac{x}{x} = 1$$

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x+y}{x-y} = \lim_{y\to 0}\frac{y}{-y} = -1$$

$\therefore\ \lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y) \neq \lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y)$ in general.

$\lim\limits_{(x,y)\to(0,0)}\dfrac{x+y}{x-y}$ DNE



2 diff. paths give diff. limits.

Remarks

(1)

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) \quad \not\Longrightarrow \quad \lim_{(x,y)\to(a,b)} f(x,y)$$

both exist & equal $\quad \not\Longleftarrow \quad$ exists (& equal)

(eg $f(x,y)=\begin{cases}1, & \text{if } x=y\\ 0, & \text{if } x\neq y\end{cases}$, $(a,b)=(0,0)$) (Ex!)

(eg, $f(x,y)=\begin{cases} x\cos\frac{1}{y}+y\cos\frac{1}{x}, & \text{if } x,y\neq 0\\ 0 & \text{if } x=0 \text{ or } y=0\end{cases}$)

$(a,b)=(0,0)$ (Ex!)

(2) If all 3 limits exist, then they are equal!

Continuity

Def: Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ & $\vec{a} \in A$ (so $f(\vec{a})$ is defined)

Then f is said to be continuous at $\vec{a}$

if $\boxed{\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})}$ (exists & equal to $f(\vec{a})$.)

Equivalently, $\forall \varepsilon>0, \exists \delta>0$ such that

if $\vec{x}\in A$ & $\|\vec{x}-\vec{a}\|<\delta$, then $|f(\vec{x})-f(\vec{a})|<\varepsilon$.

Def $f : A \to \mathbb{R}$ is said to be continuous (on A) if f is continuous at every point in A.

eg: Let $k=1,\cdots,n$. Show that $f(x_1,\cdots,x_n) = x_k$ is continuous on $\mathbb{R}^n$

(usually called the k-th coordinate function)

Pf: Let $\vec{a} = (a_1,\cdots,a_k,\cdots a_n) \in \mathbb{R}^n$

$\forall \varepsilon>0$, take $\delta=\varepsilon$.

Then, if $(0<)\|\vec{x}-\vec{a}\|<\delta$, then

$$|f(\vec{x})-f(\vec{a})| = |x_k - a_k| \le \sqrt{(x_1-a_1)^2+\cdots+(x_k-a_k)^2+\cdots+(x_n-a_n)^2}$$

$$< \delta = \varepsilon.$$

$\therefore \lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$, & hence f is continuous on $\mathbb{R}^n$.

$\#$

Thm If $f, g : A \overset{\subset \mathbb{R}^n}{} \longrightarrow \mathbb{R}$ are <u>continuous</u> at $\vec{a} \in A$, then

(1) $f(\vec{x}) \pm g(\vec{x})$, $kf(\vec{x})$, $f(\vec{x})g(\vec{x})$ are <u>continuous</u> at $\vec{a}$, where k is a constant.

(2) $\dfrac{f(\vec{x})}{g(\vec{x})}$ is <u>continuous</u> at $\vec{a}$ provided $g(\vec{a}) \neq 0$

<u>Consequences</u>:

(i) All polynomials of multi-variables are continuous (on $\mathbb{R}^n$)

(ii) All rational functions of multi-variables are continuous on their domain of definition

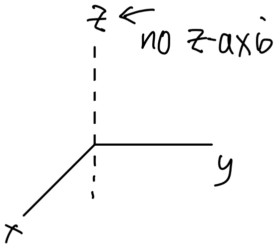
(rational function $\overset{\text{def}}{=} \dfrac{P(\vec{x})}{Q(\vec{x})}$ for some polynomials $P(\vec{x})$ & $Q(\vec{x})$

"domain of definition" of $\dfrac{P(\vec{x})}{Q(\vec{x})} = \mathbb{R}^n \setminus \{\vec{x} : Q(\vec{x}) = 0\}$)

<u>egs</u> (1) $x^3 + 3yz + z^2 - x + 7y$ is a polynomial on $\mathbb{R}^3$ & is continuous on $\mathbb{R}^3$

(2) $\dfrac{x^3 + y^2 + yz}{x^2 + y^2}$ is a rational function on $\mathbb{R}^3$

domain of definition $= \mathbb{R}^2 \setminus \{(0,0,z)\}$
$= \mathbb{R}^3 \setminus \{z\text{-axis}\}$



& is continuous on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$.

Fact: Let $\vec{a}$ be a zero of polynomial $Q(\vec{x})$ (ie. $Q(\vec{a})=0$)

Then the rational function $r(\vec{x}) = \dfrac{P(\vec{x})}{Q(\vec{x})}$ can be

"extended to a function continuous at $\vec{a}$ $\iff$ $\lim_{\vec{x}\to\vec{a}} r(\vec{x})$ exists".

egs (1) $f(x,y) = \dfrac{xy+y^3}{x^2+y^2}$ (in $\mathbb{R}^2$)

Note $x^2+y^2=0 \iff (x,y)=(0,0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy+y^3}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2+m^3x^3}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{m+m^3x}{1+m^2} = \frac{m}{1+m^2}$$

different slopes give diff. limits.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

$\therefore \dfrac{xy+y^3}{x^2+y^2}$ cannot be extended to a function continuous at $(0,0)$.

(2) $g(x,y) = \dfrac{x^4-y^4}{x^2+y^2}$ (Note $x^2+y^2=0 \iff (x,y)=(0,0)$)

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{r\to 0} \frac{r^4(\cos^4\theta - \sin^4\theta)}{r^2} = \lim_{r\to 0} r^4(\cos^4\theta - \sin^4\theta)$$

$= 0$ (by Squeeze thm)

$\therefore\ g(x,y) = \dfrac{x^4-y^4}{x^2+y^2}$ extends to a function continuous at $(0,0)$.

Infact $g(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$ $\left(g(x,y) = \dfrac{x^4-y^4}{x^2+y^2} = x^2-y^2 \right)$

<u>Thm</u> If • $f: A \subset \mathbb{R}^n \to \mathbb{R}$ is <u>continuous at $\vec{a} \in A$</u>,

• $g(x)$ is a 1-variable function <u>continuous at $f(\vec{a})$</u>

Then $g \circ f(\vec{x})$ $\left(\stackrel{\text{def}}{=} g(f(\vec{x})) \right)$ is continuous at $\vec{a}$, and

$$\lim_{\vec{x} \to \vec{a}} g(f(\vec{x})) = g\left(\lim_{\vec{x} \to \vec{a}} f(\vec{x}) \right) = g(f(\vec{a})).$$

<u>egs</u>: (1) x_k k-th coordinate function are continuous, $\forall k = 1, \cdots, n$

• $|x|$ is also continuous $\Rightarrow$ $|x_k|$ are continuous.

• $\ln|x|$ is continuous for $|x|>0$ $\Rightarrow$ $\ln|x_k|$ are continuous if $|x_k|>0$.

(2) $\sin(x^2+yz)$, e^{x-y}, $\cos\left(\frac{1}{x^2+y^2}\right)$ (except $(x,y)=(0,0)$)

$r = \sqrt{x^2+y^2}$ are continuous on their domains.

Partial Derivatives

Def: Let
- $\Omega \subseteq \mathbb{R}^n$ be open
- $f : \Omega \to \mathbb{R}$ be a function

Then the i-th partial derivative of f at $\vec{x} = (x_1, \cdots, x_n) \in \Omega$ is defined by

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \frac{\partial f}{\partial x_i}(x_1, \cdots, x_n)$$

$$= \lim_{h \to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots, x_n)}{h}$$

(provided the limit exists)

Remarks: (1) Ω open "ensures" $(x_1, \cdots, x_i + h, \cdots, x_n) \in \Omega$ for small h so that $f(x_1, \cdots, x_i + h, \cdots, x_n)$ is defined.

(2) If $n = 1$, $\frac{\partial f}{\partial x} = \frac{df}{dx}$

(3) If $n = 2$, we usually write

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

(4) In practice, $\frac{\partial f}{\partial x_i}$ can be calculated as derivative in one variable x_i by regarding other variables as constants.

(5) Other notations: $\frac{\partial f}{\partial x} = \partial_1 f = D_1 f = \nabla_1 f = f_x$

$$\frac{\partial f}{\partial y} = \partial_2 f = D_2 f = \nabla_2 f = f_y$$

eg $f(x,y) = x^2 + y^2$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x + 0 = 2x \qquad \left(\frac{\partial (y^2)}{\partial x} = 0 \right)$$

regarded as const

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 0 + 2y = 2y \qquad \left(\frac{\partial (x^2)}{\partial y} = 0 \right)$$

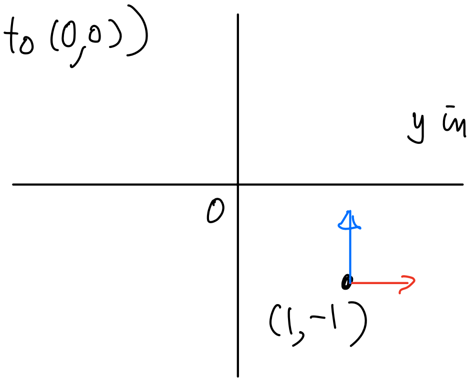
Note: As the point $(1,-1)$

$$\frac{\partial f}{\partial x}(1,-1) = 2 > 0 \quad \& \quad \frac{\partial f}{\partial y}(1,-1) = -2 < 0$$

f increases as x increases at $(1,-1)$

f decreases as y increases at $(1,-1)$

$f(x,y) = x^2 + y^2 = (\text{dist to } (0,0))^2$



y increases $\Rightarrow$ moving closer to the origin $\Rightarrow f$ decreases.

x increases $\Rightarrow$ moving away from $(0,0)$ $\Rightarrow f$ increases.

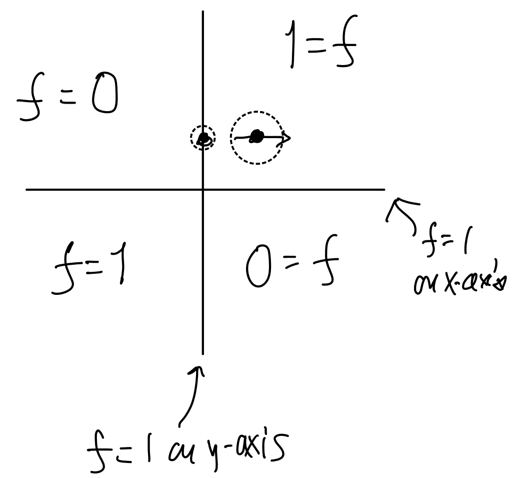
eg: $f(x,y,z) = xy^2 - \cos(xz)$

Then $f_x = y^2 + z\sin(xz)$

$f_y = 2xy$

$f_z = x\sin(xz)$

eg $f(x,y) = \begin{cases} 1 & , \text{if } xy \geq 0 \\ 0 & , \text{if } xy < 0 \end{cases}$



Find $\frac{\partial f}{\partial x}(1,1)$, $\frac{\partial f}{\partial x}(0,1)$, & $\frac{\partial f}{\partial x}(0,0)$

Soln: For $(1,1)$.

$$\lim_{h\to 0} \frac{f(1+h,1) - f(1,1)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

$\therefore \frac{\partial f}{\partial x}(1,1) = 0$.

For $(0,1)$

$$\lim_{\substack{h\to 0\\ h>0}} \frac{f(0+h,1) - f(0,1)}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{1-1}{h} = 0$$

$$\lim_{\substack{h\to 0\\ h<0}} \frac{f(0+h,1) - f(0,1)}{h} = \lim_{\substack{h\to 0\\ h<0}} \frac{0-1}{h} \text{ DNE}$$

$\therefore \frac{\partial f}{\partial x}(0,1)$ DNE.

For $(0,0)$

$$\lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0 .$$

$$\therefore \quad \frac{\partial f}{\partial x}(0,0) = 0 .$$

$\left(\frac{\partial f}{\partial x} \text{ exists at } (0,0) \not\Rightarrow f \text{ is continuous at } (0,0) \right)$

Similarly $\frac{\partial f}{\partial y}(0,0) = 0$

Higher Order Partial Derivatives

$n=2$, $f(x,y) \longrightarrow \frac{\partial f}{\partial x}(x,y), \ \frac{\partial f}{\partial y}(x,y)$ 1st order derivatives

$$\longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right), \ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

2nd order derivatives

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right), \ \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(x,y)\right)$$

be careful

Notations: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$, $\quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Of course, similarly for 3rd order derivatives: eg

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right]$$

$$= f_{yyx}$$

eg : $f(x,y) = x\sin y + y^2 e^{2x}$

Then $f_x = \sin y + 2y^2 e^{2x}$

$f_y = x\cos y + 2y e^{2x}$

$\left\{ \begin{array}{l} f_{xx} = 4y^2 e^{2x} \\ f_{xy} = \cos y + 4y e^{2x} \end{array} \right.$

$\left\{ \begin{array}{l} f_{yx} = \cos y + 4y e^{2x} \\ f_{yy} = -x\sin y + 2e^{2x} \end{array} \right.$