Limit of Multi-variable Functions

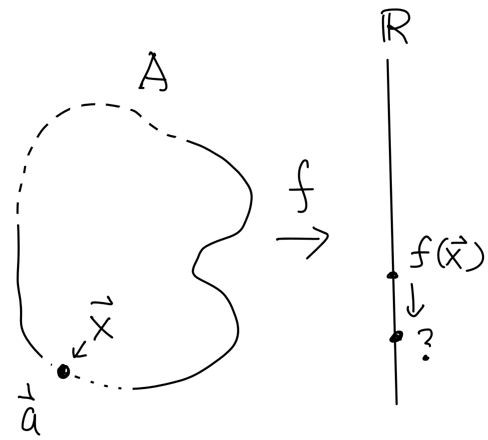
Let $A \subseteq \mathbb{R}^n$

$f : A \to \mathbb{R}$ be a function (of n-variables)

Let $\overline{A} \stackrel{\text{def}}{=\!=} A \cup \partial A$ the <u>closure</u> of A

For $\vec{a} \in A \cup \partial A$, we consider

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x})$$



In general n, m-dims

Def (ε-δ) : Let $\vec{f} : A \subset \mathbb{R}^n \to \mathbb{R}^m$, $\vec{a} \in \overline{A} = A \cup \partial A$

We say that $$\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\vec{x} \in A \text{ and } 0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow \|\vec{f}(\vec{x}) - \vec{L}\| < \varepsilon$$

<u>Remarks</u> (i) $\|\vec{x} - \vec{a}\|$ = distance between $\vec{x}$ and $\vec{a}$ in $\mathbb{R}^n$

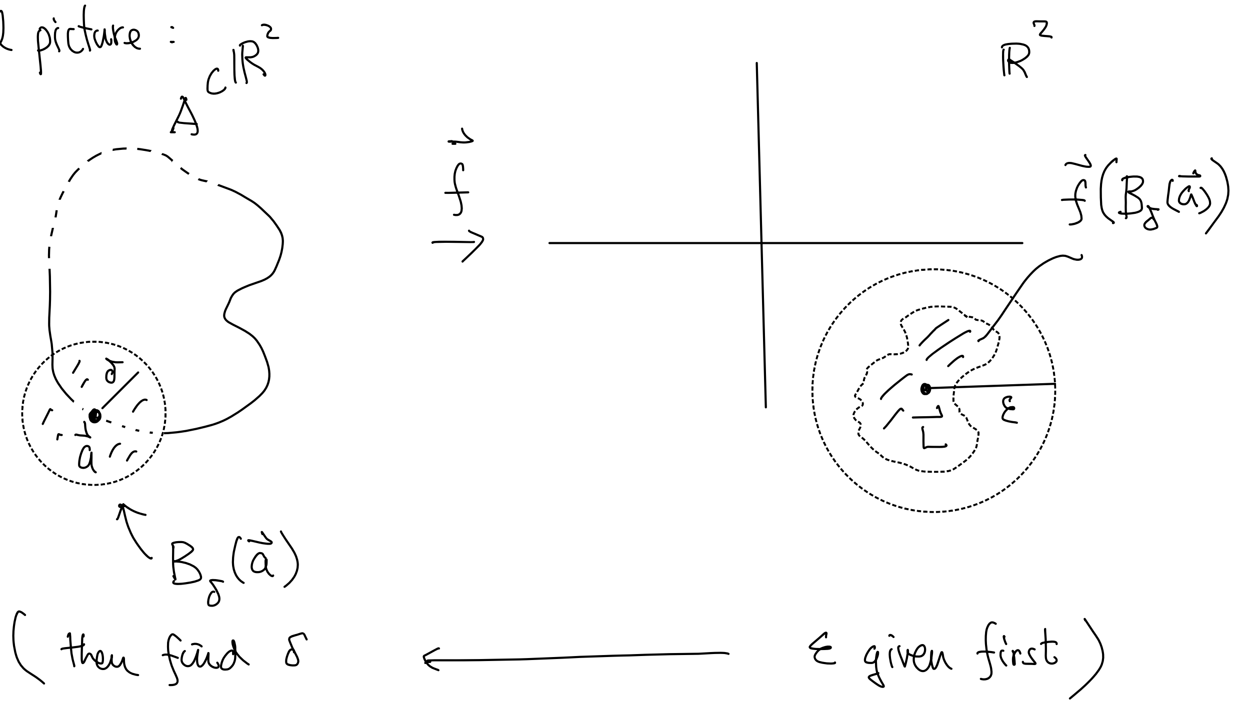
$0 < \|\vec{x} - \vec{a}\|$ means $\vec{x} \neq \vec{a}$

i.e. Considering points close to $\vec{a}$ but <u>not equal</u> to $\vec{a}$.

(ii) $\|\vec{f}(\vec{x}) - \vec{L}\|$ = distance between $\vec{f}(\vec{x})$ and $\vec{L}$ in $\mathbb{R}^m$.

If $m = 1$, $\|\vec{f}(\vec{x}) - \vec{L}\| = |f(\vec{x}) - L|$ absolute value of the different.

2 dim'l picture :



eg : $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x+y$

Illustrate that $\lim_{(x,y)\to(1,2)} f(x,y) = 3$.

Soln : [i.e. you need to show that given any $\varepsilon > 0$, one can find $\delta > 0$ such that if $0 < \|(x,y)-(1,2)\| < \delta$ then $|f(x,y) - 3| < \varepsilon$. (No need to check $(x,y) \in A$, because $A = \mathbb{R}^2$)]

Idea : $|f(x,y) - 3| = |x+y-3|$

$$= |(x-1)+(y-2)| \le |x-1| + |y-2|$$

$$\|(x,y)-(1,2)\| = \sqrt{(x-1)^2+(y-2)^2}$$

For instance, for $\varepsilon = 1$, choose $\delta = \frac{1}{2}$

if $\|(x,y)-(1,2)\| < \delta = \frac{1}{2}$, then

$$|x-1| \le \sqrt{(x-1)^2+(y-2)^2} < \frac{1}{2}$$

$$|y-2| \le \sqrt{(x-1)^2+(y-2)^2} < \frac{1}{2}$$

& hence $\quad |f(x,y)-3| \le |x-1| + |y-2| < \frac{1}{2} + \frac{1}{2} = 1 = \varepsilon$.

Similarly, for $\varepsilon = \frac{1}{100}$, one can choose $\delta = \frac{1}{200}$. (Ex!)

<u>(Real) Proof:</u> For any given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2}$. Then

$$\|(x,y)-(1,2)\| < \delta = \frac{\varepsilon}{2}$$

$$\Rightarrow |f(x,y)-3| = |x+y-3| = |(x-1)+(y-2)| \le |x-1| + |y-2|$$
$$< \delta + \delta = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(Since $|x-1| \le \|(x,y)-(1,2)\|$ & $|y-2| \le \|(x,y)-(1,2)\|$)

$$\therefore \lim_{(x,y)\to(1,2)} f(x,y) = 3 \quad ※$$

<u>eg</u>: Let $f(x,y) = x^2+y^2$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ from definition.

<u>Soln</u>: Need to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $0 < \|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$

then $\|f(x,y)-0\| = |x^2+y^2| < \varepsilon$

<u>eg</u>: $\varepsilon = \frac{1}{100}$, then $\delta = \sqrt{\varepsilon} = \frac{1}{10}$.

If $\|(x,y)-(0,0)\| < \delta = \frac{1}{10}$, then $\sqrt{x^2+y^2} < \frac{1}{10}$

$\Rightarrow \quad x^2+y^2 < \frac{1}{100}$, i.e. $\|f(x,y)-0\| < \frac{1}{100} = \varepsilon$

(Real) Proof: $\forall \varepsilon>0$, choose $\delta=\sqrt{\varepsilon}>0$.

$$0<\|(x,y)-(0,0)\|<\delta=\sqrt{\varepsilon} \Rightarrow \sqrt{x^2+y^2}<\sqrt{\varepsilon}$$

$$\Rightarrow x^2+y^2<\varepsilon \quad \text{, i.e. } \|f(x,y)-0\|=x^2+y^2<\varepsilon.$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y)=0.$$

Prop: Let
- $A\subseteq\mathbb{R}^n$
- $\vec{a}\in\overline{A}=A\cup\partial A$
- $\vec{f}:A\to\mathbb{R}^m$ with

$$\vec{f}(\vec{x})=\begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x})\end{bmatrix}=\begin{bmatrix} f_1(x_1,\cdots,x_n) \\ \vdots \\ f_m(x_1,\cdots,x_n)\end{bmatrix}$$

where $\vec{x}=(x_1,\cdots,x_n)\in A$.

Then

$$\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})=\vec{L}=\begin{bmatrix} l_1 \\ \vdots \\ l_m\end{bmatrix} \Leftrightarrow \lim_{\vec{x}\to\vec{a}} f_i(\vec{x})=l_i, \ \forall i=1,\cdots,m.$$

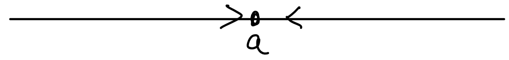
Consequence: It is good enough for us to focus on limit of real-valued function $f:A\to\mathbb{R}$ (ie. $m=1$)

eg: $\vec{f}:\mathbb{R}^2\to\mathbb{R}^2$ $\vec{f}(x,y)=\begin{bmatrix} x+y \\ x^2+y^2+1\end{bmatrix}$

$$\lim_{(x,y)\to(1,2)} \vec{f}(x,y)=\begin{bmatrix} \lim_{(x,y)\to(1,2)}(x+y) \\ \lim_{(x,y)\to(1,2)}(x^2+y^2+1)\end{bmatrix}=\begin{bmatrix} 3 \\ 6\end{bmatrix} (\leftarrow \text{Ex!})$$

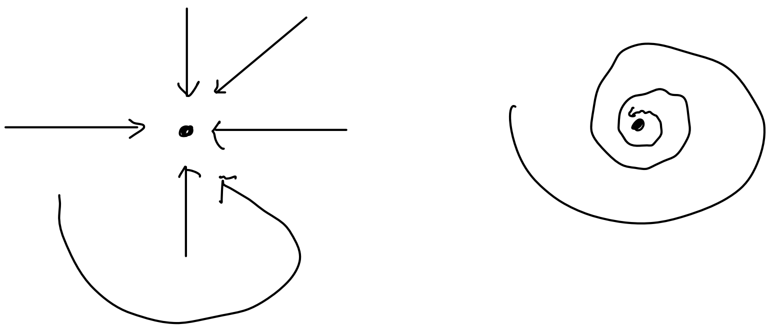
Limit along a path

Recall: In one variable:



$$\lim_{x\to a} f(x) \text{ exists} \Leftrightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$
(exist & equal)

For n-variables, $n \geq 2$, there are infinitely many ways to approach a point in $\mathbb{R}^n$. Situation is very complicated.



However, we still have the following

> Fact: $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $\vec{a} \in \bar{A} = A \cup \partial A$. Then
>
> $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L \Leftrightarrow$ limit of $f(\vec{x})$ when $\vec{x}$ approaches to $\vec{a}$ along <u>any curve (path)</u> exists and equals to L

- Useful for showing <u>limit</u> "<u>does not exists</u>" (DNE) (only in our dept. not a common notation)

(i) Find a path such that the limit along that path <u>DNE</u>, or

(ii) Find 2 paths such that the limits along the 2 paths <u>are different</u>

$\Rightarrow \lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})$ DNE.

eg $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ $\left(\dfrac{x^2-y^2}{x^2+y^2} \text{ doesn't define at } (x,y)=(0,0) \right)$

Soln: (1) Along x-axis $(y=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0} 1 = 1$$

(2) Along y-axis $(x=0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0} -1 = -1$$

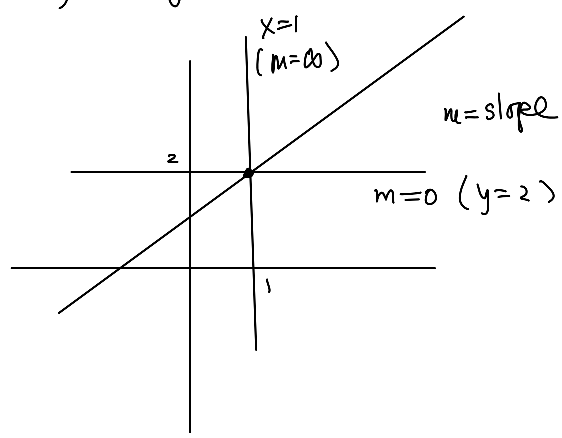
$\therefore$ Different limits along different paths

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ DNE.}$$

(In fact, we can try other paths too, for instance, $y=x$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} \frac{x^2-y^2}{x^2+y^2} = 0 \quad (\text{Ex!})$$
)

eg: Consider $\lim\limits_{(x,y)\to(1,2)} \dfrac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$ along all straight lines passing through $(1,2)$.



Soln: (1) Along $x=1$.

$$\lim_{\substack{(x,y)\to(1,2)\\ x=1}} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$$

$$= \lim_{y\to 2} \frac{(1)y-2(1)-y+2}{(1-1)^2+(y-2)^2} = \lim_{y\to 0} 0 = 0$$

(2) Along the line with slope $= m$ & passing through $(1,2)$:

$$y-2 = m(x-1)$$

$$\lim_{\substack{(x,y)\to(1,2) \\ y-2=m(x-1)}} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2} = \lim_{\substack{(x,y)\to(1,2) \\ y-2=m(x-1)}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} \quad \text{(check!)}$$

$$= \lim_{x\to 1} \frac{m(x-1)^2}{(x-1)^2+(m(x-1))^2} = \lim_{x\to 1} \frac{m}{1+m^2}$$

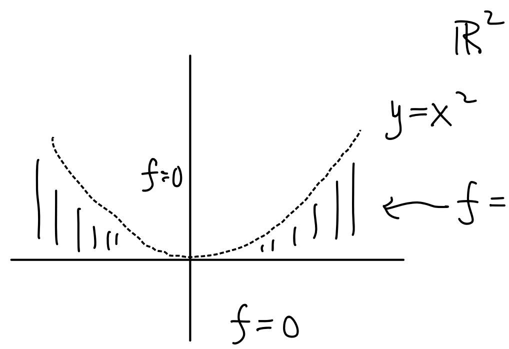
$$= \frac{m}{1+m^2}$$

(Different limits for different slopes (ie different paths)

$\therefore \lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$ DNE.)

eg: $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1, & \text{if } 0<y<x^2 \\ 0, & \text{otherwise} \end{cases}$$



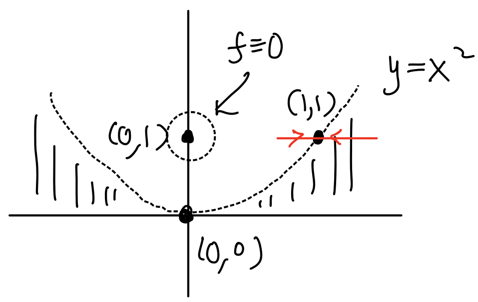
($f\equiv 1$ for |||...|||, 0 other places)

Find $\lim_{(x,y)\to\vec{a}} f(x,y)$, where

(i) $\vec{a}=(0,1)$

(ii) $\vec{a}=(1,1)$

(iii) $\vec{a}=(0,0)$



(i) For $\vec{a}=(0,1)$, $f(x,y)\equiv 0$ near $(0,1)$ $\Rightarrow \lim\limits_{(x,y)\to(0,1)} f(x,y)=0$

(ii) For $\vec{a}=(1,1)$,

$$\lim_{\substack{(x,y)\to(1,1)\\ x<1,\, y=1}} f(x,y) = \lim_{\substack{x\to 1^-\\ x<1,\, y=1}} 0 = 0$$

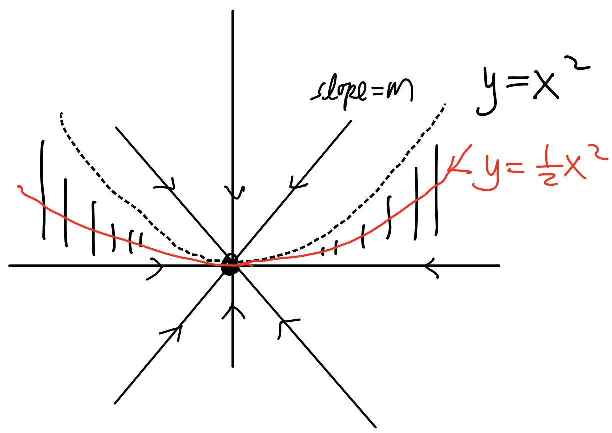
$$\#$$

$$\lim_{\substack{(x,y)\to(1,1)\\ x>1,\, y=1}} f(x,y) = \lim_{\substack{x\to 1^+\\ x>1,\, y=1}} 1 = 1$$

Different limits for different paths,

$\therefore \lim\limits_{(x,y)\to(1,1)} f(x,y)$ DNE.

(iii) <u>Case 1</u> Along y-axis $(x=0)$



$f(0,y)=0,\ \forall y$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y)=0$$

<u>Case 2</u>: Along $y=mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{x\to 0} f(x,mx) = 0$$

<u>Case 3</u> Along the curve $y=\frac{1}{2}x^2$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\frac{1}{2}x^2}} f(x,y) = \lim_{x\to 0} f(x,\tfrac{1}{2}x^2) = \lim_{x\to 0} 1 \quad \left(f(x,\tfrac{1}{2}x^2)=1 \ \ \forall x\neq 0 \right)$$

$$= 1$$

$\therefore$ Case 3 & Case 2 together $\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE. $\#$

<u>Properties of Limits</u>

Assuming all limits on the right hand side exist, then the limit on the left hand side exists and the formula holds

(1) $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm \lim_{\vec{x}\to\vec{a}} g(\vec{x})$

(2) $\lim_{\vec{x}\to\vec{a}} k f(\vec{x}) = k \lim_{\vec{x}\to\vec{a}} f(\vec{x})$, where k is a constant

(3) $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \lim_{\vec{x}\to\vec{a}} g(\vec{x})$

(4) $\lim_{\vec{x}\to\vec{a}} \dfrac{f(\vec{x})}{g(\vec{x})} = \dfrac{\lim_{\vec{x}\to\vec{a}} f(\vec{x})}{\lim_{\vec{x}\to\vec{a}} g(\vec{x})}$ if $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) \neq 0$

(5) $\lim_{\vec{x}\to\vec{a}} \left[f(\vec{x})\right]^n = \left[\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right]^n$, $n \geq 0$ (integer)

(6) $\lim_{\vec{x}\to\vec{a}} \left[f(\vec{x})\right]^{\frac{1}{n}} = \left[\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right]^{\frac{1}{n}}$, $n \geq 0$ (integer)
(If n is even, assume $f(\vec{x}) \geq$ near $\vec{a}$.)

Squeeze Theorem (Sandwich Theorem)

Let $f, g, h : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be functions of n-variables

If $\begin{cases} \bullet\ g(\vec{x}) \leq f(\vec{x}) \leq h(\vec{x}) \text{ near } \vec{a} \in \Omega \text{ and} \\ \bullet\ \lim\limits_{\vec{x} \to \vec{a}} g(\vec{x}) = \lim\limits_{\vec{x} \to \vec{a}} h(\vec{x}) = L. \end{cases}$

Then $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = L$

Special Case of Squeeze Theorem

If $\begin{cases} \bullet\ |f(\vec{x})| \leq g(\vec{x}) \text{ near } \vec{a} \text{ and} \\ \bullet\ \lim\limits_{\vec{x} \to \vec{a}} g(\vec{x}) = 0 \end{cases}$

Then $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = 0.$

$(|f| \leq g \Rightarrow -g \leq f \leq g)$

eg: $\lim\limits_{(x,y) \to (0,0)} x \cos\left(\frac{1}{x^2+y^2}\right)$

Soln: $\left|\cos\left(\frac{1}{x^2+y^2}\right)\right| \leq 1 \Rightarrow \left|x\cos\left(\frac{1}{x^2+y^2}\right)\right| \leq |x|$

Also $\lim\limits_{(x,y) \to (0,0)} |x| = 0.$

Sequeeze Thm $\Rightarrow \lim\limits_{(x,y) \to (0,0)} x\cos\left(\frac{1}{x^2+y^2}\right) = 0.$ ※

eg: $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}$ ($\ln x$ = natural log = $\log x$)

Soln: $\left|\frac{(x-1)^2\ln x}{(x-1)^2+y^2}\right| = \left|\frac{(x-1)^2}{(x-1)^2+y^2}\right| |\ln x| \leq |\ln x|$

Also $\displaystyle\lim_{(x,y)\to(1,0)} |\ln x| = 0$

$\therefore$ Squeeze Thm $\Rightarrow$ $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2} = 0$. #