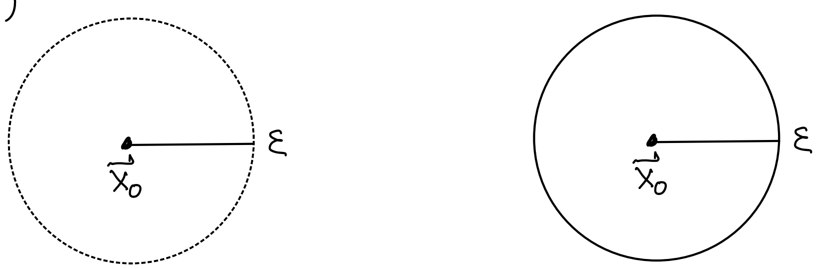
Topological Terminology in $\mathbb{R}^n$

Def
- $B_\varepsilon(\vec{x}_0) = \{ \vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| < \varepsilon \}$ is called the <u>open ball</u> of radius ε and centered at $\vec{x}_0$
- $\overline{B_\varepsilon(\vec{x}_0)} = \{ \vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| \leq \varepsilon \}$ is called the <u>closed ball</u> of radius ε and centered at $\vec{x}_0$

<u>Remark</u>: If $n=2$, $B_\varepsilon(\vec{x}_0)$, $\overline{B_\varepsilon(\vec{x}_0)}$ are referred as <u>open disk</u>, <u>closed disk</u> and denoted by $D_\varepsilon(\vec{x}_0)$, $\overline{D_\varepsilon(\vec{x}_0)}$ in some textbooks.

$B_\varepsilon(\vec{x}_0)$ $\qquad\qquad\qquad$ $\overline{B_\varepsilon(\vec{x}_0)}$



points on the dotted "line" are not included

points on the solid "line" are included

<u>Recall notation</u>: " $\exists$ " : there exist(s)

" $\forall$ " : for all

Def: Let S be a set in $\mathbb{R}^n$.

(1) The interior of S is the set

$$\mathrm{Int}(S) = \{\vec{x} \in \mathbb{R}^n : \exists\, \varepsilon > 0 \text{ s.t. } B_\varepsilon(\vec{x}) \subset S\}$$

Points in $\mathrm{Int}(S)$ are called interior points of S

(2) The exterior of S is the set

$$\mathrm{Ext}(S) = \{\vec{x} \in \mathbb{R}^n : \exists\, \varepsilon > 0 \text{ s.t. } B_\varepsilon(\vec{x}) \subset \mathbb{R}^n \setminus S\}$$

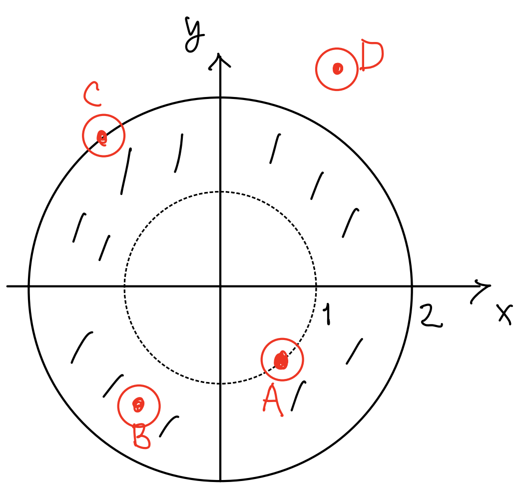
Points in $\mathrm{Ext}(S)$ are called exterior points of S

(3) The boundary of S is the set

$$\partial S = \left\{\vec{x} \in \mathbb{R}^n : \forall\, \varepsilon > 0 \text{ s.t. } \begin{array}{l} B_\varepsilon(\vec{x}) \cap S \neq \phi, \ \& \\ B_\varepsilon(\vec{x}) \cap (\mathbb{R}^n \setminus S) \neq \phi \end{array}\right\}$$

Points on ∂S are called boundary points of S

eg: $S = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4\} \subset \mathbb{R}^2$



A = boundary point
B = interior point
C = boundary point
D = exterior point

$$\mathrm{Int}(S) = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$$

$$\mathrm{Ext}(S) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \text{ or } x^2 + y^2 > 4\}$$

$$\partial S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ or } x^2 + y^2 = 4\}$$

Prop Let $S \subset \mathbb{R}^n$. Then

(1) $\mathbb{R}^n$ = disjoint union of $\mathrm{Int}(S)$, $\mathrm{Ext}(S)$ and ∂S

(2) $\mathrm{Int}(S) \subseteq S$

$\mathrm{Ext}(S) \subseteq \mathbb{R}^n \setminus S$

(3) A point on ∂S may or may not be in S

(For statement (3), see points A & C in the above eg.)

Def A set $S \subset \mathbb{R}^n$ is called

(1) <u>open</u> if $\forall x \in S$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$

(2) <u>closed</u> if $\mathbb{R}^n \setminus S$ is open

<u>Equivalent definition</u>:

(1) S open $\Leftrightarrow$ $S = \mathrm{Int}(S)$

(2) S closed $\Leftrightarrow$ $S = \mathrm{Int}(S) \cup \partial S$

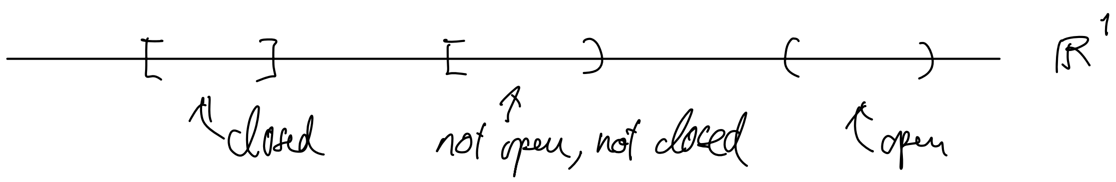
(check!)

<u>eg</u> Is $S = \{ (x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4 \}$ open or closed?

<u>Answer</u>: Not open, and

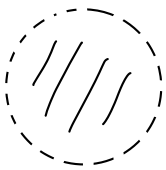
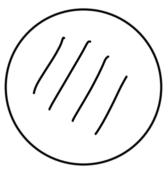
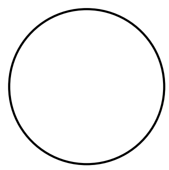
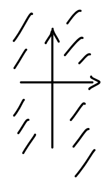
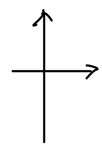
Not closed!

(Similar to



)

Eg:

Subset $S \subset \mathbb{R}^2$	$B_1(0,0) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$	$\overline{B_1(0,0)} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$	$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$	$\mathbb{R}^2$	ϕ (empty set)
$\text{Int}(S)$	$B_1(0,0)$	$B_1(0,0)$	ϕ	$\mathbb{R}^2$	ϕ
$\text{Ext}(S)$	$\mathbb{R}^2 \setminus \overline{B_1(0,0)} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2>1\}$	$\mathbb{R}^2 \setminus \overline{B_1(0,0)}$	$\mathbb{R}^2 \setminus S^1$	ϕ	$\mathbb{R}^2$
∂S	S^1	S^1	S^1	ϕ	ϕ
Open?	Yes	No	No	Yes	Yes
Closed?	No	Yes	Yes	Yes	Yes
Picture					

<u>Remarks</u>: (1) There are exactly two subsets of $\mathbb{R}^n$ which are <u>both</u> open and closed: $\mathbb{R}^n$ and ϕ.

(2) Some subsets of $\mathbb{R}^n$ are <u>neither</u> open <u>nor</u> closed. (eg: above)

(3) For any $S \subseteq \mathbb{R}^n$, $\text{Int}(S)$ & $\text{Ext}(S)$ are <u>open</u> in $\mathbb{R}^n$; ∂S is <u>closed</u> in $\mathbb{R}^n$

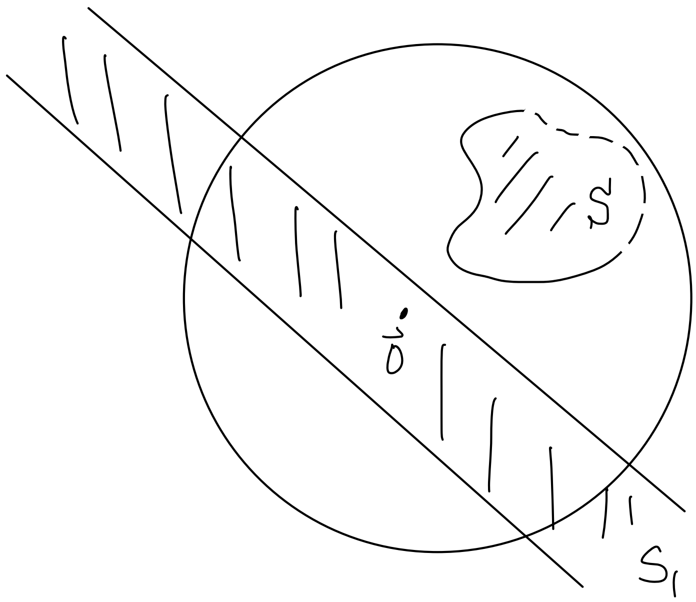
(Ex: What about $\text{Int}(S) \cup \partial S$?)

Def: $S \subseteq \mathbb{R}^n$ is called <u>bounded</u> if

$\exists M > 0$ such that

$$S \subseteq B_M(\vec{0}) = \{ \vec{x} \in \mathbb{R}^n : \|\vec{x}\| < M \}$$

S is called <u>unbounded</u> if it is not bounded



S is <u>bounded</u>

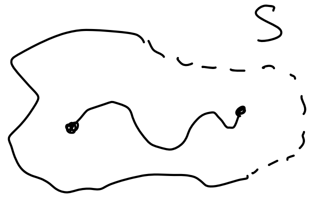
S_1 is <u>unbounded</u>

eg: $y\text{-axis} = \{ (x,y) \in \mathbb{R}^2 : x = 0 \}$ is unbounded
(Pf: $\forall M > 0$, $\exists (0, 2M) \in y\text{-axis}$ s.t. $(0, 2M) \notin B_M(\vec{0})$.)

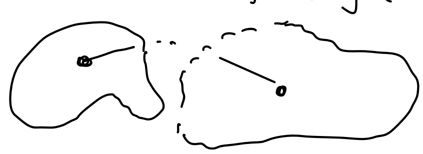
Def $S \subset \mathbb{R}^n$ is called <u>path-connected</u> if any two points in S can be connected by a curve in S.

S

path-connected

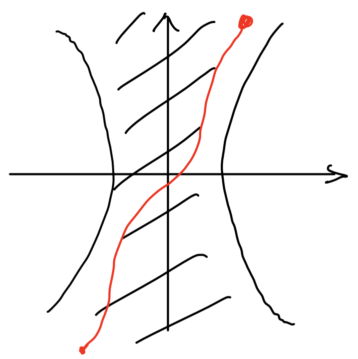


cannot joint by a curve completely inside S.

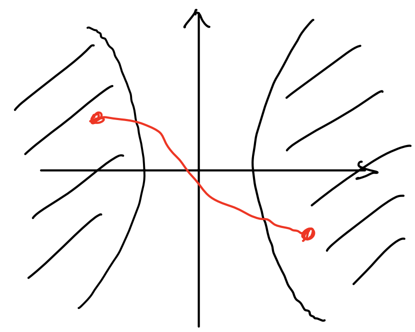


eg: $S = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 \leq 1\}$ is path-connected

$S_1 = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 \geq 1\}$ is not path-connected.



S



S_1

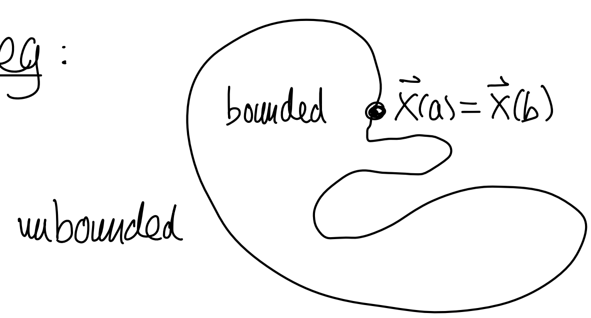
<u>Remark</u>: In topology, there is a different notion called "connected". We'll not discuss it.

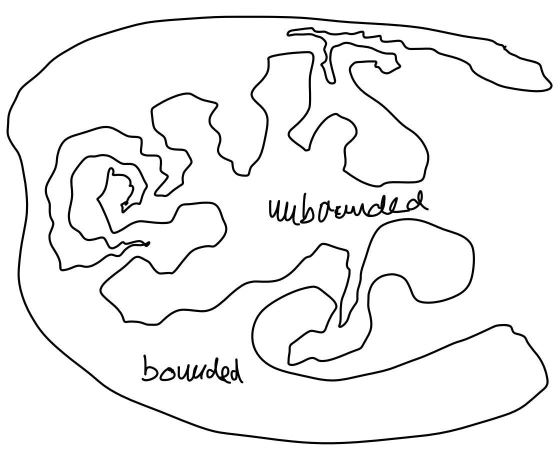
<u>Thm</u> (<u>Jordan Curve Theorem</u>)

A <u>simple</u> <u>closed</u> curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into 2 <u>path-connected</u> components, with one <u>bounded</u> and one <u>unbounded</u>

<u>Remark</u>: "closed curve" means continuous curve $\vec{x}(t)$, $a \leq t \leq b$ with $\vec{x}(a) = \vec{x}(b)$. And one can show that it is a "closed subset" in $\mathbb{R}^2$.

eg:





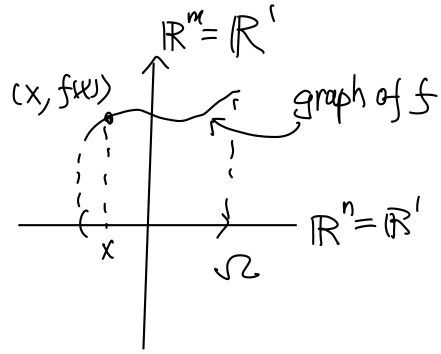
Vector-valued functions of Multivariables

$\vec{f} : \Omega^{\subset \mathbb{R}^n} \longrightarrow \mathbb{R}^m$ How to visualize it?

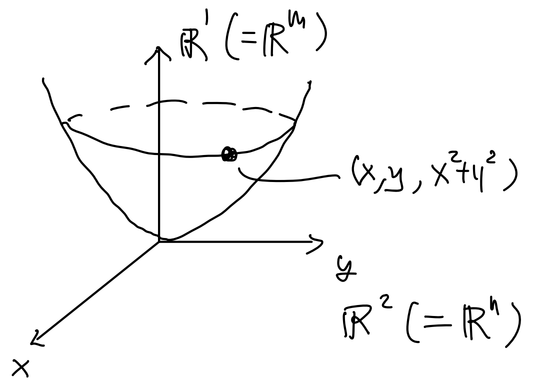
(1) Graph of $\vec{f}$

$$\mathrm{Graph}(\vec{f}) = \{ (\underbrace{\vec{x}}_{\in \mathbb{R}^n}, \underbrace{\vec{f}(\vec{x})}_{\in \mathbb{R}^m}) : \vec{x} \in \Omega \} \subseteq \mathbb{R}^{n+m}$$

$n = m = 1$



eg: $n=2$, $m=1$: $g : \mathbb{R}^2 \to \mathbb{R}$ defined $g(x,y) = x^2+y^2$



graph(g) is the surface
$= \{ (x, y, x^2+y^2) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2 \} \subseteq \mathbb{R}^3$

(In general, it is impossible to draw the graph for $n+m > 3$!)

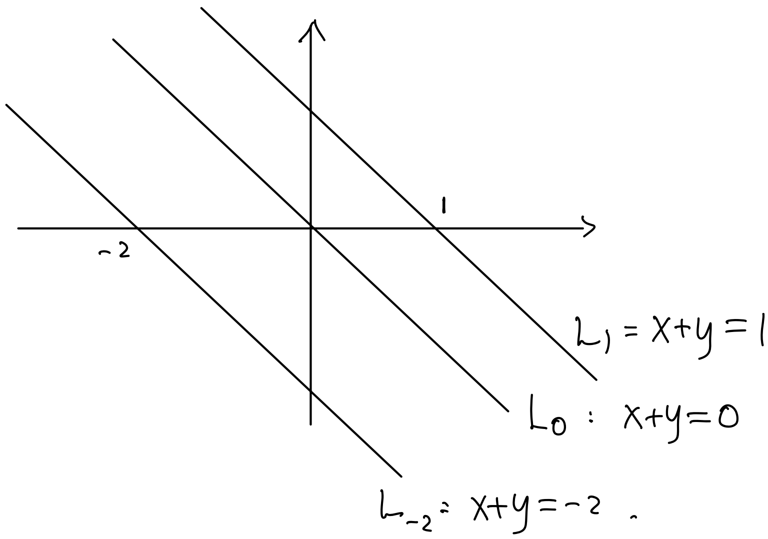
(2) Level set of $\vec{f} : \Omega^{\subset \mathbb{R}^n} \longrightarrow \mathbb{R}^m$

If $\vec{c} \in \mathbb{R}^m$, define the level set at $\vec{c}$ to be

$$L_{\vec{c}} = \{ \vec{x} \in \Omega \subset \mathbb{R}^n : \vec{f}(\vec{x}) = \vec{c} \} = (\vec{f})^{-1}(\vec{c}) \subseteq \Omega \subseteq \mathbb{R}^n.$$

eg: $f(x,y)=x+y$, $\Omega=\mathbb{R}^2$ ($m=1 \Rightarrow \vec{c}\in\mathbb{R}^1$, ie $\vec{c}$ is a number)

$$L_c=\{(x,y)\in\mathbb{R}^2: f(x,y)=c\}$$
$$=\{(x,y)\in\mathbb{R}^2: x+y=c\}$$



eg: $g(x,y)=x^2+y^2$, $\Omega=\mathbb{R}^2$

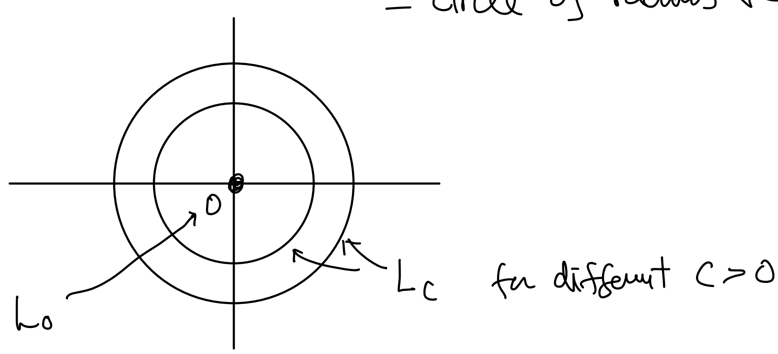
$$L_c=\{(x,y)\in\mathbb{R}^2: g(x,y)=c\}$$
$$=\{(x,y)\in\mathbb{R}^2: x^2+y^2=c\}$$

<u>Case 1</u>: $c<0$, $L_c=\emptyset$

<u>Case 2</u>: $c=0$, $L_0=\{(0,0)\}$

<u>Case 3</u>: $c>0$, $L_c=\{(x,y)\in\mathbb{R}^2: x^2+y^2=c\}$

$=$ circle of radius $\sqrt{c}$ centered at $(0,0)$.



eg: $h(x,y) = \cos(2\pi(x^2+y^2))$, $\Omega = \mathbb{R}^2$

$$L_c = \{(x,y) \in \mathbb{R}^2 : h(x,y) = c\}$$

$$= \{(x,y) \in \mathbb{R}^2 : \cos(2\pi(x^2+y^2)) = c\}$$

<u>Case 1</u>: If $|c| > 1$, then $L_c = \phi$.

<u>Case 2</u>: If $|c| \le 1$, then $L_c = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = \frac{1}{2\pi}\cos^{-1}(c)\}$

need $\cos^{-1}(c) \ge 0$.

<u>subcase (a)</u> $\cos^{-1}(c) < 0$

$L_c = \phi$

<u>Subcase (b)</u> $\cos^{-1}(c) = 0$

$L_c = \{(0,0)\}$

<u>subcase (c)</u> $\cos^{-1}(c) > 0$

$L_c =$ circle of radius $\sqrt{\frac{1}{2\pi}\cos^{-1}(c)}$ centered at $(0,0)$.

<u>Level set $\leftrightarrow$ graph</u>