Arclength (of a curve)

Let $\vec{X}(t)$ be a curve with $\vec{X}'(t)$ exists and continuous

Def: Arclength of $\vec{X}(t)$ for $a \leq t \leq b$ is

$$s = \int_a^b \|\vec{X}'(t)\|\, dt$$

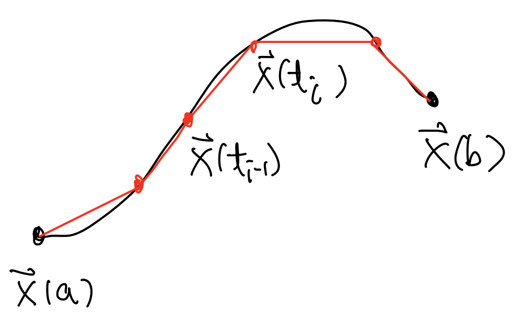
Remark: If $\vec{X}(t)$ = displacement at time t

then $\vec{X}'(t)$ = velocity

$\|\vec{X}'(t)\|$ = speed

$$\int_a^b \|\vec{X}'(t)\|\, dt = \text{distance travelled.}$$

Idea of the defn (from mathematician's point of view)



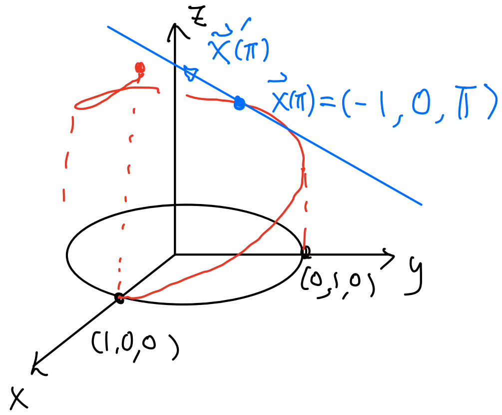
approximate the curve by straight line segments.

$$s \approx \sum_i \|\vec{X}(t_i) - \vec{X}(t_{i-1})\| \qquad \left(\vec{X}'(t_i) = \lim_{t \to t_i} \frac{\vec{X}(t) - \vec{X}(t_i)}{t - t_i} \right)$$

$$\approx \sum_i \|\vec{X}'(t_i)\|(t_i - t_{i-1}) \longrightarrow \int_a^b \|\vec{X}'(t)\|\, dt$$

"as the approximation get better & better" eg: (Helix)

$\vec{X}(t) = (\cos t, \sin t, t)$ (curve in $\mathbb{R}^3$) $\quad t \in [0, 2\pi]$



(a) Find the tangent line of $\vec{X}$ at $t = \pi$

(b) Find arclength of the helix

(a): $\vec{X}(t) = (\cos t, \sin t, t)$

$\vec{X}'(t) = (-\sin t, \cos t, 1)$

$\therefore$ $\vec{X}(\pi) = (-1, 0, \pi)$ is a point on the tangent line

$\vec{X}'(\pi) = (0, -1, 1)$ is a direction vector of the tangent line.

$\therefore$ The tangent line at $t = \pi$ is given by

$$\vec{y}(\tau) = \vec{X}(\pi) + \tau \vec{X}'(\pi) \qquad (\tau \in \mathbb{R})$$

$$= (-1, 0, \pi) + \tau(0, -1, 1)$$

(b) $\|\vec{X}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$

$\Rightarrow$ arclength $s = \int_0^{2\pi} \|\vec{X}'(t)\| \, dt = \int_0^{2\pi} \sqrt{2} \, dt$

$$= 2\sqrt{2}\pi$$

Remark: Arclength is independent of change of parameters! (Proof omitted) (Ex)

Polar Coordinates in $\mathbb{R}^2$

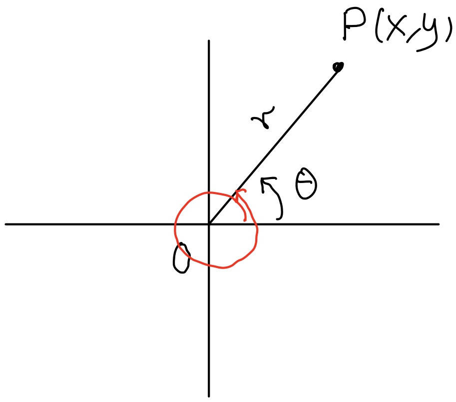
$P=(x,y)\in\mathbb{R}^2$ can be represented by

$(r\cos\theta,\ r\sin\theta)$

where $r=\sqrt{x^2+y^2}$ = distance from origin

θ = angle from the positive x-axis to $\overrightarrow{OP}$

in counter-clockwise direction



Remarks: (i) $(r\cos\theta, r\sin\theta) = (r\cos(\theta+2k\pi),\ r\sin(\theta+2k\pi))$

for any $k\in\mathbb{Z}=\{\cdots,-2,-1,0,1,2,\cdots\}$

(ii) For $P=(0,0)$, then $\begin{cases} r=0 \\ \theta \text{ is not well-defined}\end{cases}$

(iii) For our defn., we usually set

$r\in[0,\infty)$ $\quad (r\geq 0)$

$\theta\in[0,2\pi)$ $\quad (0\leq\theta<2\pi)$

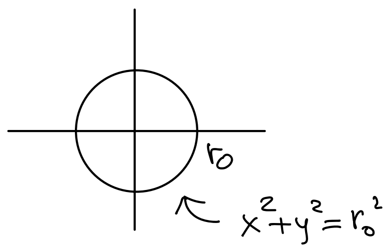
But in some book, $\begin{cases} r\in\mathbb{R} \text{ (can be negative as in Textbook)} \\ \text{(see later example)} \\ \theta\in\mathbb{R}\end{cases}$

Change of Coordinates formula

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \quad (\text{for } x>0)$$

Curves in Polar Coordinates

eg : Circle of radius $r_0 > 0$, centered at origin



Polar Equation :

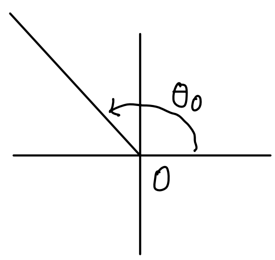
$$r = r_0 .$$

Parametric Form (in polar)

$$\begin{cases} r = r_0 \\ \theta = t , \quad t \in [0, 2\pi] \end{cases}$$

$$\left(\vec{x}(t) = (r_0\cos t, r_0\sin t) \right)$$

eg Half ray from origin



Polar Equation

$$\theta = \theta_0$$

Parametric form (in polar)

$$\begin{cases} r = t , \quad t \in [0, \infty) \\ \theta = \theta_0 \end{cases}$$

eg: <u>Archimedes Spiral</u>

$k>0$ is a constant.

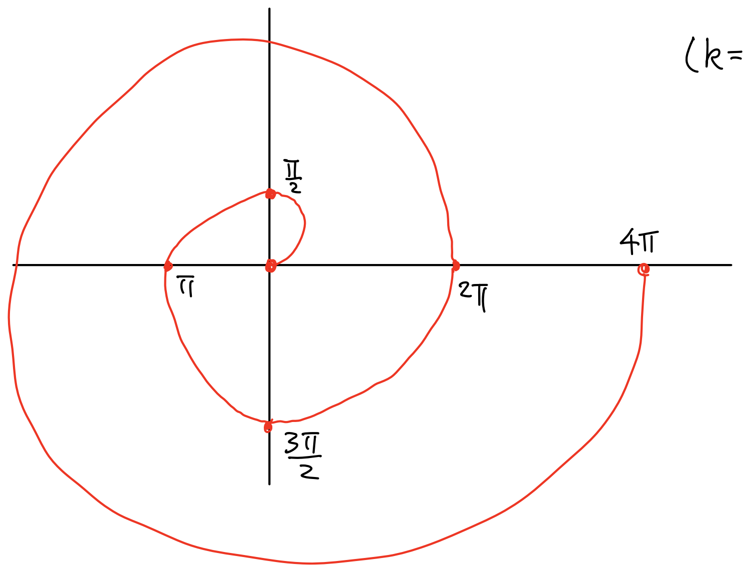
<u>Polar Equation</u>

$r = k\theta$

<u>Parametric form</u> (in pole)

$$\begin{cases} r = kt \\ \theta = t \end{cases} \qquad t \in [0,\infty)$$

(k=1)



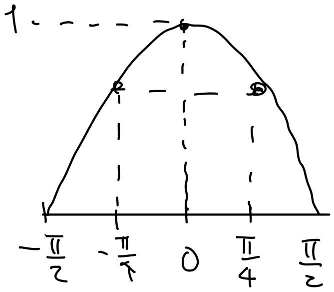
<u>Remark</u>: $\vec{x}(t) = (r\cos\theta, r\sin\theta) = (kt\cos t, kt\sin t) = k(t\cos t, t\sin t)$

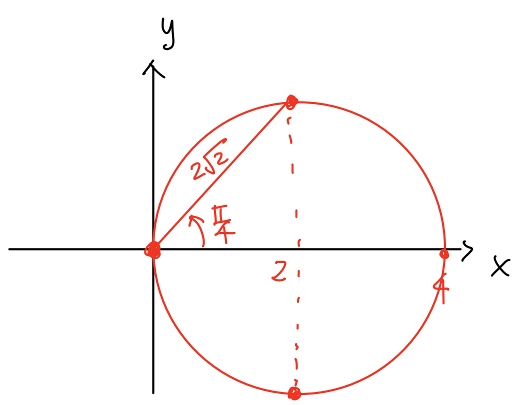
$\Rightarrow$ $\vec{x}'(t) = k(\cos t - t\sin t, \sin t + t\cos t)$ is the tangent vector

at $\vec{x}(t) = k(t\cos t, t\sin t)$.

$\left((r'(t), \theta'(t)) = (k, 1) \text{ is not the tangent vector in } \mathbb{R}^2 \right)$

eg: $r = 4\cos\theta$ $\left(\geq 0, \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right)$





$\cos\theta \geq 0$

$r = 4\cos\theta \Rightarrow r^2 = 4r\cos\theta$

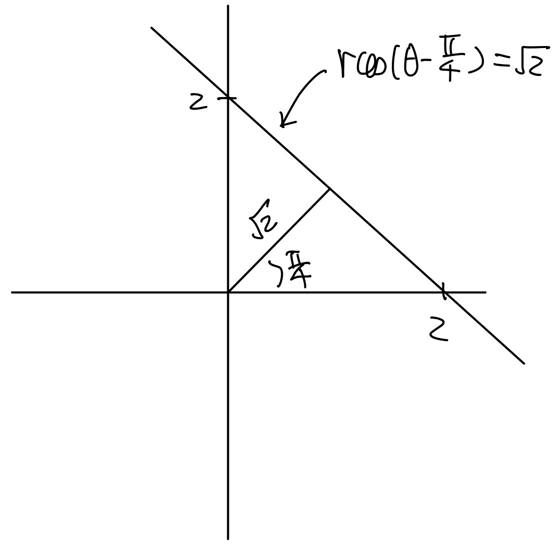
$\Rightarrow x^2+y^2 = 4x$

$\Rightarrow (x-2)^2+y^2 = 2^2$ a circle of radius 2 centered at $(2,0)$.

eg: $r\cos(\theta - \frac{\pi}{4}) = \sqrt{2}$

$$\begin{aligned} &\| \\ &r\left(\cos\theta\cos\tfrac{\pi}{4} + \sin\theta\sin\tfrac{\pi}{4}\right) \\ &\| \\ &\frac{r\cos\theta}{\sqrt{2}} + \frac{r\sin\theta}{\sqrt{2}} \\ &\| \\ &\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{aligned}$$

$\therefore\ x+y = 2$



Negative r

Our convention is $r \geq 0$.

But sometimes is convenient to allow $r<0$ by the interpretation

$$\begin{aligned} (x,y) &= (r\cos\theta, r\sin\theta) \\ &= (-|r|\cos\theta, -|r|\sin\theta) \\ &= -(|r|\cos\theta, |r|\sin\theta) \qquad \left(= (|r|\cos(\theta+\pi), |r|\sin(\theta+\pi))\right) \end{aligned}$$

eg: $r=-2$, $\theta = \frac{\pi}{6}$ $\quad (x,y) = (-2\cos\frac{\pi}{6}, -2\sin\frac{\pi}{6}) = -(\sqrt{3},1) = (-\sqrt{3},-1)$

eg: $r = 1-(1+\varepsilon)\cos\theta$, $\varepsilon>0$

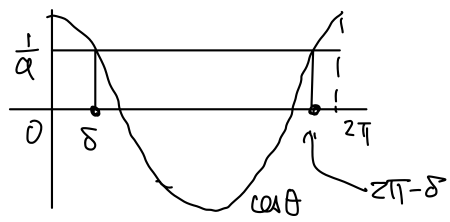
$= 1 - a\cos\theta$, $a = 1+\varepsilon > 1$

Case 1 : $r \geq 0 \Rightarrow 1 - a\cos\theta \geq 0 \Rightarrow \cos\theta \leq \frac{1}{a} < 1$

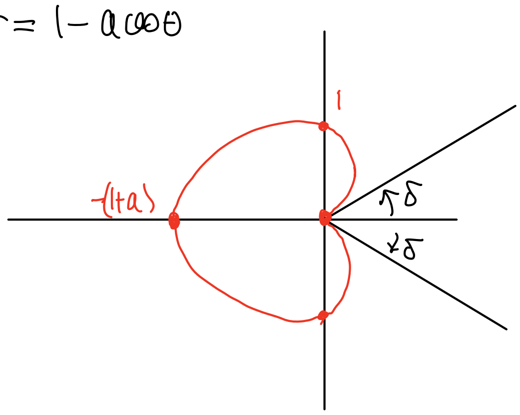
$\Rightarrow$ θ cannot run through the whole interval $[0, 2\pi]$

but only $[\delta, 2\pi-\delta]$

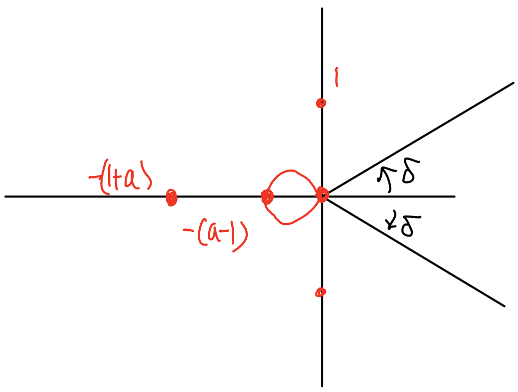
where $\delta = \cos^{-1}\left(\frac{1}{a}\right)$



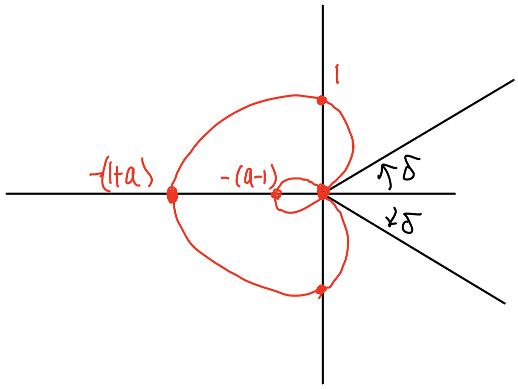
$r = 1 - a\cos\theta$



Case 2 $r<0$ $0 \leq \theta < \delta$, $2\pi - \delta < \theta \leq 2\pi$



So if we allow $r \in \mathbb{R}$, then $r = 1 - a\cos\theta$ can be defined for all $\theta \in [0, 2\pi]$ & the curve becomes a curve with self-intersection.



Coordinates Systems in $\mathbb{R}^3$ (generalizing polar coordinates)

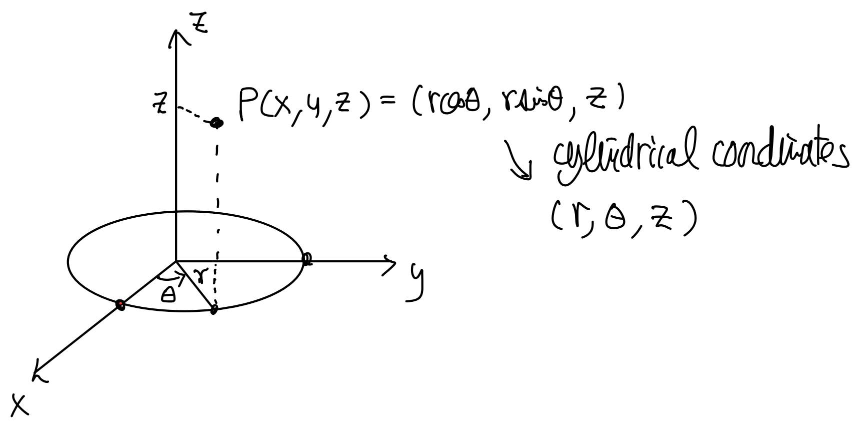
Cylindrical Coordinates

$$(x, y, z) \longrightarrow (r, \theta, z)$$

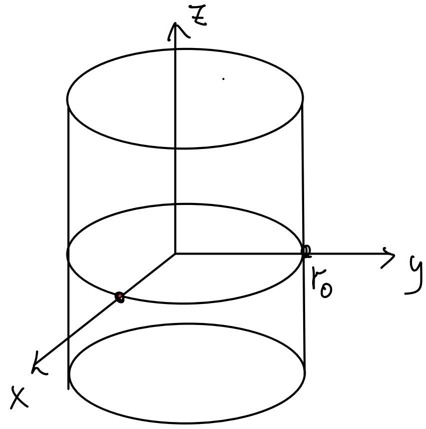
where $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ i.e. (r, θ) is polar coordinates for xy-plane.

Formula

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



eg: $r = r_0$ constant is the equation for the cylinder in the figure.



eg <u>Helix</u> $(a\cos t, a\sin t, bt)$ $(a>0,\ b\in\mathbb{R})$

can be represented in cylindrical coordinates by

$$\begin{cases} r = a \\ \theta = t \\ z = bt \end{cases} \qquad t\in[0,2\pi]$$

(Ex: Sketch the curve)

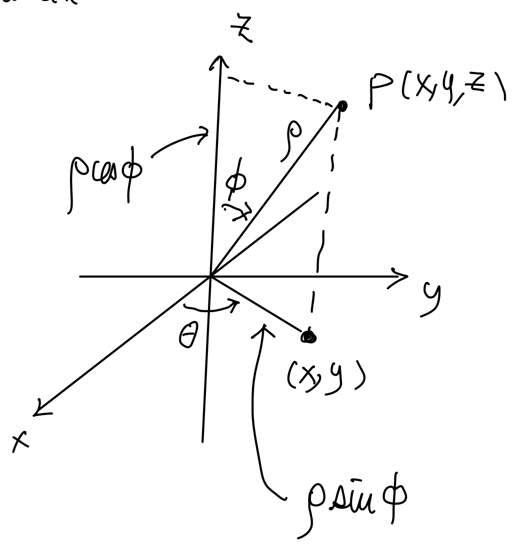
Spherical Coordinates

$P=(x,y,z)\in\mathbb{R}^3$ can be represented by

ρ = distance from origin $=\sqrt{x^2+y^2+z^2}$

θ = θ as in cylindrical coordinates

ϕ = angle from positive z-axis to $\overrightarrow{OP}$.



<u>Remark</u>: $\phi\in[0,\pi]$

<u>Formulae</u>

$$\boxed{\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}}$$

(Tutorial for Egs)