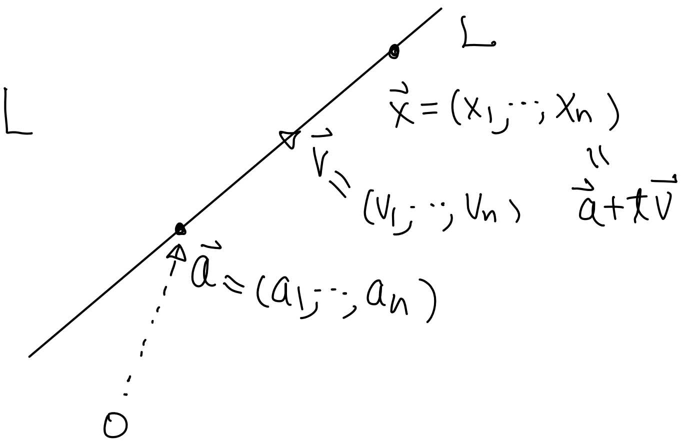
# Parametric Form of a line in $\mathbb{R}^n$ ($n=3$ particular)

Let $L$ = a line in $\mathbb{R}^n$

$\vec{a}$ = a point on $L$

$\vec{v}$ = a direction vector of $L$




Then

**parametric form of L**

$$\vec{x} = \vec{a} + t\vec{v}$$

$t \in \mathbb{R}$ called a parameter

($L$ is parametrized by $t \in \mathbb{R}$)

i.e.

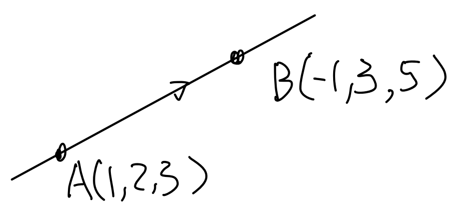
$$(x_1, \cdots, x_n) = (a_1, \cdots, a_n) + t(v_1, \cdots, v_n)$$
$$= (a_1 + tv_1, \cdots, a_n + tv_n)$$

i.e.

$$\begin{cases} x_1 = a_1 + tv_1 \\ \vdots \\ x_n = a_n + tv_n \end{cases} \quad t \in \mathbb{R}$$

eg A line $L$ in $\mathbb{R}^3$ passes through

$A = (1,2,3)$, $B = (-1,3,5)$




Soln:

(Choose $\vec{a}$ = A or B as vector
$\vec{v} = \overrightarrow{AB}$ or $\overrightarrow{BA}$ )

A parametrization of $L$ is $\vec{x} = (1,2,3) + t\big((-1,3,5) - (1,2,3)\big)$

$$= (1,2,3) + t(-2,1,2) \quad \text{——} (*)$$

(In high school notations: $x = 1-2t$, $y = 2+t$, $z = 3+2t$ )

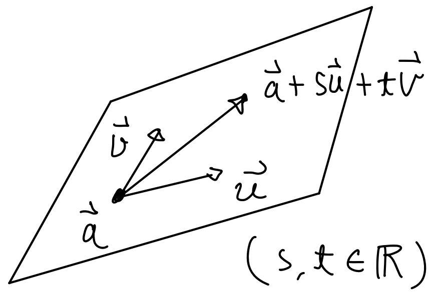
Remarks: (i) Parametric form is not unique (many choice as in eg)

(ii) From (*), we get symmetric form:

$$\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-3}{2} \quad (=t)$$

$$\Leftrightarrow \begin{cases} x-1 = -2(y-2) \\ 2(y-2) = z-3 \end{cases}$$

## Planes in $\mathbb{R}^3$




(1) $P$ = a plane in $\mathbb{R}^3$

$\vec{a}$ = a point on $P$

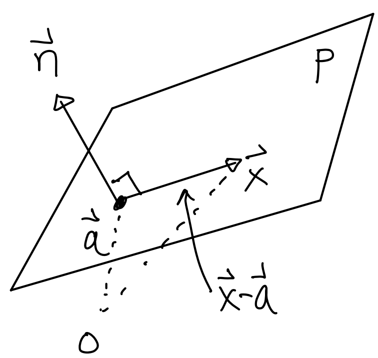
$\vec{u}, \vec{v}$ = 2 linearly independent vectors on $P$.

Then

**Parametric Form of P**

$$\vec{x} = \vec{a} + s\vec{u} + t\vec{v}$$

two parameters




(2) $P$ = a plane in $\mathbb{R}^3$

$\vec{a}$ = a point on $P$

$\vec{n}$ = a normal vector of $P$

(i.e. $\vec{n}$ is perpendicular to $P$)

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{n} = (n_1, n_2, n_3)$ and $\vec{x} = (x, y, z)$

$\vec{x} \in P \Leftrightarrow (\vec{x} - \vec{a}) \perp \vec{n}$

$\Leftrightarrow (\vec{x}-\vec{a})\cdot\vec{n}=0 \quad (\Leftrightarrow \vec{x}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

$\Leftrightarrow (x-a_1, y-a_2, z-a_3)\cdot(n_1,n_2,n_3)=0$

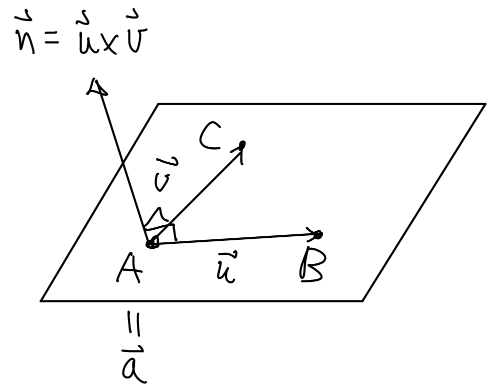
$\Leftrightarrow n_1x+n_2y+n_3z=\underbrace{n_1a_1+n_2a_2+n_3a_3}_{\text{constant}}.$

$$\boxed{\begin{array}{c}\underline{\text{Equation of } P} \text{ (general in } \mathbb{R}^3)\\ n_1x+n_2y+n_3z=c\end{array}}$$

$(c=\vec{a}\cdot\vec{n})$
provided $\vec{n}=(n_1,n_2,n_3)\neq\vec{0}$.

eg: Suppose $P$ is a plane (in $\mathbb{R}^3$) passing through
$A=(0,0,1)$, $B=(0,2,0)$, $C=(-1,1,0)$
Represent $P$ using (i) parametric form; (ii) equation.

Soln: (i) $\left(\begin{array}{l}\text{Pick } \vec{a}=A, B \text{ or } C\\ \vec{u},\vec{v}=\vec{AB} \,\&\, \vec{AC} \text{ or } \cdots\end{array}\right)$




$\vec{AB}=(0,2,0)-(0,0,1)=(0,2,-1)$
$\vec{AC}=(-1,1,0)-(0,0,1)=(-1,1,-1)$

Then a parametric form of $P$ is

$$\vec{x}=(0,0,1)+s(0,2,-1)+t(-1,1,-1) \quad (s,t\in\mathbb{R})$$

(ii) Take $\vec{n}=\vec{u}\times\vec{v}$ ($\vec{u},\vec{v}$ as in (i))

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 2 & -1\\ -1 & 1 & -1\end{vmatrix}=(-1,1,2) \qquad \text{(check!)}$$

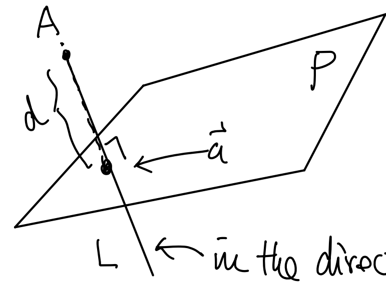
$\therefore$ Equation of $P$: $\big((x,y,z)-(0,0,1)\big)\cdot(-1,1,2)=0$

$\cdots \Leftrightarrow -x+y+2z=2 \qquad$ (check!)

eg: Find the distance between

$A = (2,1,1)$ and the

$P: \ -x + 2y - z = -4 \quad \text{---} (*)$




in the direction of $(-1,2,-1)$ (by the understanding of the equation of $P$.)

Soln: From $(*)$

$$\vec{n} = (-1,2,-1) \perp P$$

Consider the line $L$ (passing $A$ & in the direction of $\vec{n}$)

$$\vec{x} = \vec{A} + t\vec{n} = (2,1,1) + t(-1,2,-1) = (2-t, \ 1+2t, \ 1-t)$$

Then the intersection point $\vec{a}$ of $L$ & $P$ is the point closest to $A$

To find $\vec{a}$, put $\vec{x} = (2-t, \ 1+2t, \ 1-t)$

into $(*)$ $\quad -(2-t) + 2(1+2t) - (1-t) = -4$

$$\Rightarrow \quad t = -\tfrac{1}{2} \quad \text{(check!)}$$

$$\therefore \quad \vec{a} = \left(2-\left(-\tfrac{1}{2}\right), 1+2\left(-\tfrac{1}{2}\right), 1-\left(-\tfrac{1}{2}\right)\right) = \left(\tfrac{5}{2}, 0, \tfrac{3}{2}\right)$$

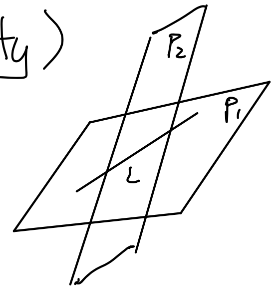
$\therefore$ distance between $A$ & $P$ = distance between $A$ & $\vec{a}$

$$= \sqrt{\left(2-\tfrac{5}{2}\right)^2 + (1-0)^2 + \left(1-\tfrac{3}{2}\right)^2}$$

$$= \frac{\sqrt{6}}{2} \quad \text{(check!)}$$

eg: <u>Line in $\mathbb{R}^3$ by equations</u>

Two planes intersect at a line (if not empty)




$$\begin{cases} x+y+6z=6 \\ x-y-2z=-2 \end{cases} \qquad \left( \begin{array}{c} (1,1,6)\ \&\ (1,-1,-2) \\ \text{are linearly indep.} \end{array} \right)$$

is a line. Then Gaussian Elimination will give us a parametric form of the line. i.e solving the system of linear equation by setting a variable to be a parameter: eg

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix} \qquad (\text{by setting } z=t) \quad (\text{linear algebra!})$$

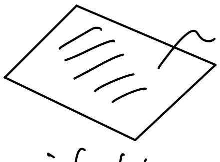
eg: How about 3 linear equations?

Then Linear Alg $\Rightarrow$ Case 1: unique solution, i.e. intersection = {point}.

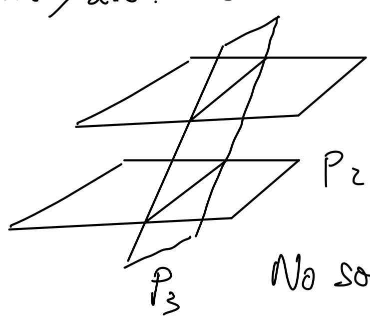
Case 2: Infinitely many solutions; could be a line or a plane

Case 3: No solution, i.e. no intersection.

eg.




infinitely many solutions




No solution

(Ex: Try other situations)

**Remark**: In $n$ dim., a (hyper)plane is given by $\vec{X} \cdot \vec{n} = c$.
as in planes in $\mathbb{R}^3$ $\quad$ ( dim (hyperplane) $= n-1$ )

Then linear algebra $\Rightarrow$ all possible situations for intersections of (hyper)planes.

(Discussion omitted)

## Curves in $\mathbb{R}^n$

**Defn**: Let $I \subset \mathbb{R}$ be an interval. A (continuous) curve in $\mathbb{R}^n$ is a continuous (vector-valued) function

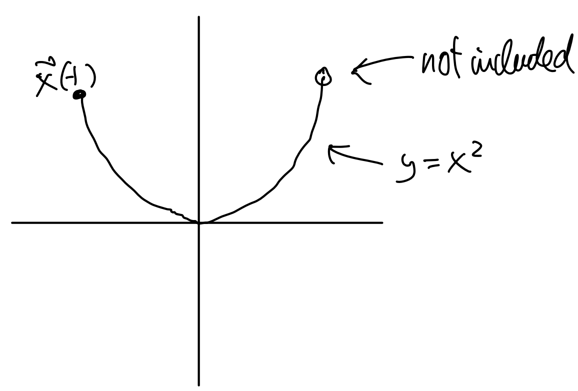
$$\vec{X} : I \longrightarrow \mathbb{R}^n$$

i.e. $t \in I$, $\vec{X}(t) = (X_1(t), \cdots, X_n(t)) \in \mathbb{R}^n$ such that every component function $X_i(t)$ is continuous $(i = 1, \cdots, n)$

egs (i) $\vec{X} : [-1, 1) \to \mathbb{R}^2$

$\vec{X}(t) = (t, t^2)$

$[\ X = t, y = t^2 \Rightarrow y = X^2\ ]$




(ii) Of course, parametric form of a line gives a "curve" $\quad \vec{X}(t) = \vec{p} + t\vec{q} \quad t \in (-\infty, \infty)$

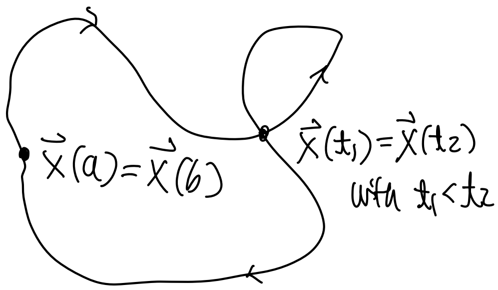
**Defn:** A curve $\vec{x}:[a,b]\to\mathbb{R}^n$ is said to be

(i) <u>closed</u> if $\vec{x}(a)=\vec{x}(b)$

(ii) <u>simple</u> if $\vec{x}(t_1)\neq\vec{x}(t_2)$ for $a\le t_1<t_2\le b$

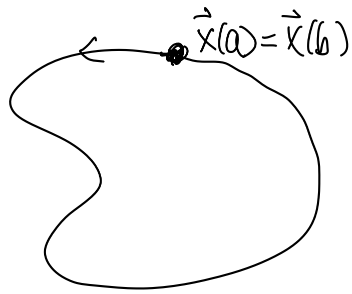
except possibly at $t_1=a$ & $t_2=b$.

<u>eg</u>:




<u>closed</u>, <u>not simple</u>


<u>not closed</u>, <u>simple</u>




<u>Closed & simple</u>

( simple closed curve )

**Thm:** Let $\vec{x}(t)=(x_1(t),\cdots,x_n(t))$. Then

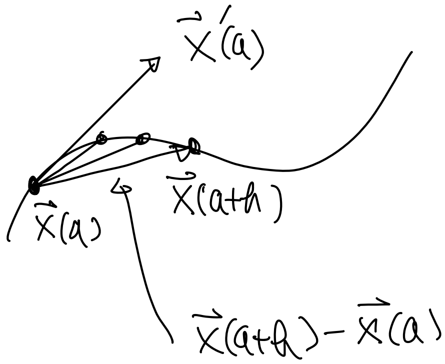
(1) $\lim\limits_{t\to a}\vec{x}(t)=\left(\lim\limits_{t\to a}x_1(t),\cdots,\lim\limits_{t\to a}x_n(t)\right)$

(2) $\vec{x}'(t)\overset{\text{def}}{=\!=}\lim\limits_{h\to 0}\dfrac{\vec{x}(t+h)-\vec{x}(t)}{h}=(x_1'(t),\cdots,x_n'(t))$

(provided limits exist)

$\boxed{\underline{\text{Defn}}: \quad \vec{x}'(a) = \underline{\text{tangent vector}} \text{ of } \vec{x}(t) \text{ at } t = a.}$

Picture:




Physics: If $\vec{x}(t) = \underline{\text{displacement}}$ at time $t$

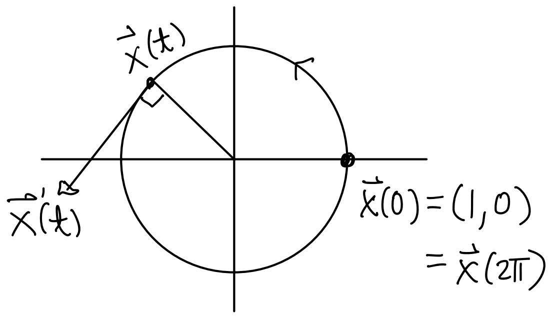
Then $\vec{x}'(t) = \underline{\text{velocity (vector)}}$ at time $t$

$\vec{x}''(t) = \underline{\text{acceleration (vector)}}$

$\|\vec{x}'(t)\| = \text{speed}$.

eg: $\vec{x}(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$

( $\overset{\parallel}{x}$ $\overset{\parallel}{y}$ $\Rightarrow x^2 + y^2 = 1$ the $\underline{\text{unit circle}}$ )




(simple closed)

$\vec{x}'(t) = (-\sin t, \cos t)$

is the tangent vector.

( $\vec{v}$ = velocity

$= \vec{x}'(t) = (-\sin t, \cos t)$

$\vec{a}$ = acceleration

$= \vec{x}''(t) = (-\cos t, -\sin t)$

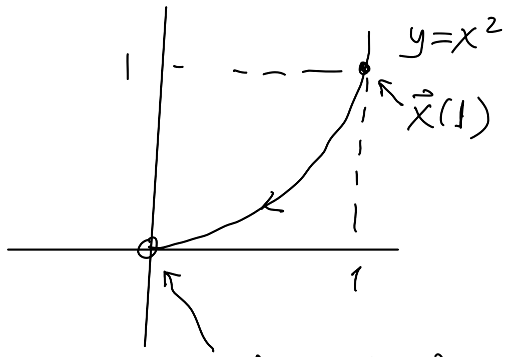
$= -\vec{x}(t)$ )

speed $= \|\vec{x}'(t)\| = 1$

eg $\vec{x} : [1, \infty) \to \mathbb{R}^2$

$\vec{x}(t) = \left(\frac{1}{t}, \frac{1}{t^2}\right)$

$\left(\begin{matrix} \parallel & \parallel \\ x & y \end{matrix} \Rightarrow y = x^2\right)$




$\lim_{t\to\infty} \vec{x}(t) = (0,0)$

(not included in the curve)

<u>Rules</u>

Let $\vec{x}(t)$, $\vec{y}(t)$ be curves in $\mathbb{R}^n$, $c \in \mathbb{R}$ be a constant, $f(t)$ be a real-valued function. Then

(1) $\left(\vec{x}(t) + \vec{y}(t)\right)' = \vec{x}'(t) + \vec{y}'(t)$

(2) $\left(c\,\vec{x}(t)\right)' = c\,\vec{x}'(t)$

(3) $\left(f(t)\,\vec{x}(t)\right)' = f'(t)\,\vec{x}(t) + f(t)\,\vec{x}'(t)$

(4) $\left(\vec{x}(t) \cdot \vec{y}(t)\right)' = \vec{x}'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot \vec{y}'(t)$

(5) For $n=3$,

$$\left(\vec{x}(t) \times \vec{y}(t)\right)' = \vec{x}'(t) \times \vec{y}(t) + \vec{x}(t) \times \vec{y}'(t)$$

<u>Remark</u>: (3), (4) & (5) are all called <u>product rules</u>.