<u>Remark</u>: Cauchy-Schwarz inequality $\Rightarrow$

$$-1 \le \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} \le 1 \qquad (\text{provided } \vec{a}\ne\vec{0},\ \vec{b}\ne\vec{0})$$

$\Rightarrow$ The formula $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$ defining the angle

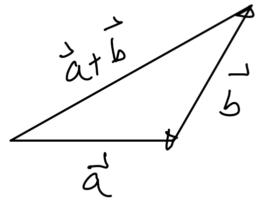
between $\vec{a}$ & $\vec{b}$ (in dim. $n \ge 4$) is well-defined.

(If $n \le 3$, we've proved the formula)

<u>Triangle Inequality</u>

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then

$$\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$$



Equality holds $\Leftrightarrow$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \ge 0$

<u>Pf</u>: $\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$= \|\vec{a}\|^2 + 2\,\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$

(Cauchy-Schwarz) $\le \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2$

$= (\|\vec{a}\| + \|\vec{b}\|)^2$

$\Rightarrow \|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$.

Equality holds $\Leftrightarrow$ $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|$ $\Rightarrow$ Equality holds for Cauchy-Schwarz.

$\Rightarrow$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$

Putting back, $\Rightarrow$ $\left.\begin{array}{l} r\|\vec{b}\|^2 = \|\vec{a}\|\|\vec{b}\| \\ \text{or } r\|\vec{a}\|^2 = \|\vec{a}\|\|\vec{b}\| \end{array}\right\}$ $\Rightarrow$ $r \ge 0$ (provided $\vec{a}\ne\vec{0}$ or $\vec{b}\ne\vec{0}$)

If $\vec{a}=\vec{b}=\vec{0}$, the statement is trivially correct. #

Option Ex: Cauchy-Schwarz inequality $\Leftrightarrow$ Triangle Inequality.

(" $\Rightarrow$ " done, " $\Leftarrow$ " Ex.)

<u>Special structure of $\mathbb{R}^3$</u>: <u>Cross Product</u> $\vec{a}\times\vec{b}$

Let $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)\in\mathbb{R}^3$

Then the <u>cross product</u> $\vec{a}\times\vec{b}$ is defined by

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$$

$$= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

where $\hat{i}=(1,0,0)$, $\hat{j}=(0,1,0)$, $\hat{k}=(0,0,1)$.

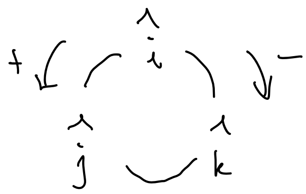
<u>eg</u>: Let $\vec{a}=(2,3,5)$ & $\vec{b}=(1,2,3)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\hat{k} = -\hat{i}-\hat{j}+\hat{k}$$

$$= (-1,-1,1)$$

Remark

$$\boxed{\begin{array}{lll} \hat{i}\times\hat{i}=\vec{0} & \hat{i}\times\hat{j}=\hat{k} & \hat{i}\times\hat{k}=-\hat{j} \\ \hat{j}\times\hat{i}=-\hat{k} & \hat{j}\times\hat{j}=\vec{0} & \hat{j}\times\hat{k}=\hat{i} \\ \hat{k}\times\hat{i}=\hat{j} & \hat{k}\times\hat{j}=-\hat{i} & \hat{k}\times\hat{k}=\vec{0} \end{array}}$$ (check!)



Properties of Cross Product

Let $\vec{a},\vec{b},\vec{c}\in\mathbb{R}^3$, $\alpha,\beta\in\mathbb{R}$

Algebraic ((1) & (2) follow from properties of determinant)

(1) $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

(2) $(\alpha\vec{a}+\beta\vec{b})\times\vec{c}=\alpha\,\vec{a}\times\vec{c}+\beta\,\vec{b}\times\vec{c}$

$\vec{a}\times(\alpha\vec{b}+\beta\vec{c})=\alpha\,\vec{a}\times\vec{b}+\beta\,\vec{a}\times\vec{c}$

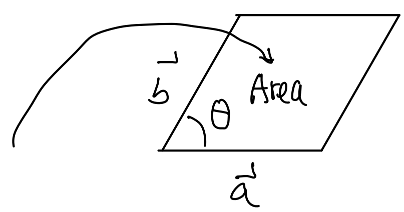
(3) $(\vec{a}\times\vec{b})\cdot\vec{a}=(\vec{a}\times\vec{b})\cdot\vec{b}=0$ (easy from definition)

Geometric

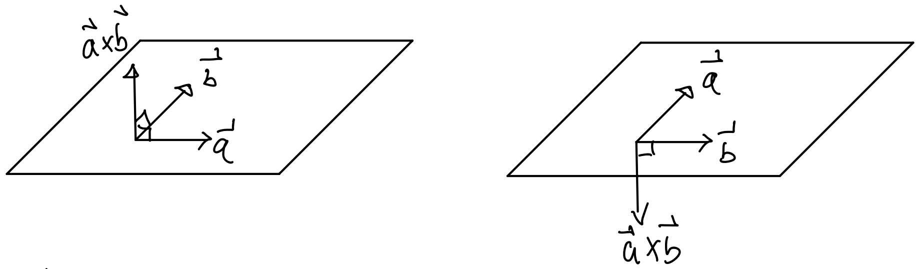
(4) Let $\theta=$ angle between $\vec{a}$ & $\vec{b}$, then

$\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|\sin\theta$

= Area of the parallelogram spanned by $\vec{a}$ & $\vec{b}$



<u>Remarks</u> (i) Formula (3) $\Rightarrow$ $\vec{a}\times\vec{b} \perp \vec{a}$ & $\vec{a}\times\vec{b} \perp \vec{b}$



(Also $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$ satisfy right-hand rule by checking the defn.)

(ii) Formula (4) : $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \text{Area}(\text{▱}) = 0$

$\Leftrightarrow \vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$

$\Leftrightarrow \{\vec{a}, \vec{b}\}$ is linearly dependent

(linear algebra)

<u>Pf</u> of (4) :

By straight forward calculation (expanding both sides using defn.):

$$\|\vec{a}\times\vec{b}\|^2 = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\right)^2 + \left(-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\right)^2 + \left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)^2$$

$$= \cdots \qquad (\text{Ex})$$

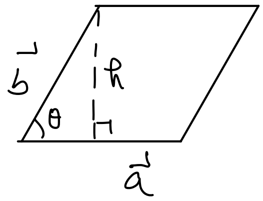
$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= \|\vec{a}\|^2 \|\vec{b}\|^2 - (\|\vec{a}\|\|\vec{b}\|\cos\theta)^2$$

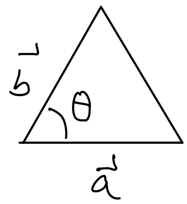
$$= \|\vec{a}\|^2 \|\vec{b}\|^2 (1 - \cos^2\theta)$$

$$= (\|\vec{a}\|\|\vec{b}\|\sin\theta)^2 .$$



$h = \|\vec{b}\| \sin\theta$

$$\Rightarrow \text{Area} = \|\vec{a}\|\, h = \|\vec{a}\|\left(\|\vec{b}\|\sin\theta\right)$$
$$= \|\vec{a}\times\vec{b}\|$$

<u>Remarks:</u> (i) Area of 

$= \frac{1}{2}\|\vec{a}\times\vec{b}\|$

(ii) $\vec{a}, \vec{b} \in \mathbb{R}^2$, i.e. $\vec{a} = (a_1, a_2, 0)$
$\vec{b} = (b_1, b_2, 0)$

$$\vec{a}\times\vec{b} = \left(0, 0, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)$$

$$\Rightarrow \text{Area}(\text{▱}) = \left| \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right|$$ absolute value of the 2x2 determinant

$$= \left| \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \right|$$

<u>Triple Product</u> (only in $\mathbb{R}^3$)

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$

The <u>triple product</u> of $\vec{a}, \vec{b}$ & $\vec{c}$ (order is important) is defined by $(\vec{a}\times\vec{b})\cdot\vec{c}$

<u>Note</u> $$(\vec{a}\times\vec{b})\cdot\vec{c} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)\cdot(c_1, c_2, c_3)$$

$$= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

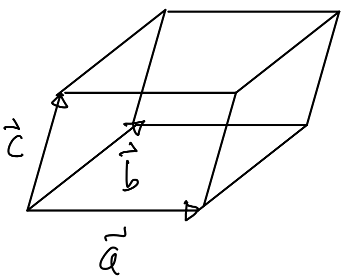
$$\boxed{(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

<u>Remark</u>: It is easy to obtain

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b} \qquad (\text{Ex})$$
$$= -(\vec{b} \times \vec{a}) \cdot \vec{c} = -(\vec{a} \times \vec{c}) \cdot \vec{b} = -(\vec{c} \times \vec{b}) \cdot \vec{a}$$

<u>Geometric meaning</u>

$\left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right| =$ Volume of the parallelopiped spanned by $\vec{a}, \vec{b}$ & $\vec{c}$.



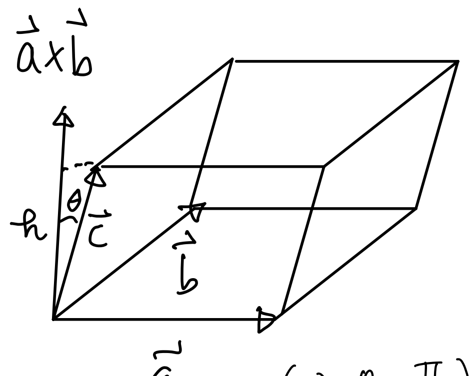
<u>Pf</u>: $h = \|\vec{c}\| \cos\theta$

and $(\vec{a} \times \vec{b}) \cdot \vec{c}$

$= \|\vec{a} \times \vec{b}\| \, \|\vec{c}\| \cos\theta$

$=$ Area$\left(\text{parallelogram spanned by } \vec{a}, \vec{b}\right) \cdot h$

$=$ Volume of the parallelopiped.

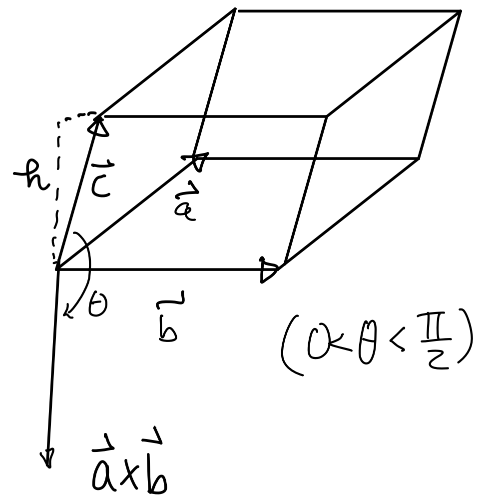


$(0 < \theta < \frac{\pi}{2})$

For case: $\frac{\pi}{2} < \theta \le \pi$ is similar

$(\vec{a} \times \vec{b}) \cdot \vec{c}$

$= -$ Volume of the parallelepiped



$(0 < \theta < \frac{\pi}{2})$

$\Rightarrow$

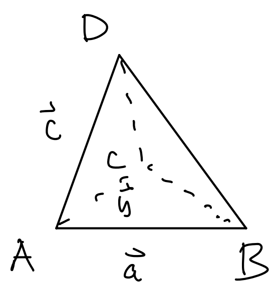
$|(\vec{a} \times \vec{b}) \cdot \vec{c}|$

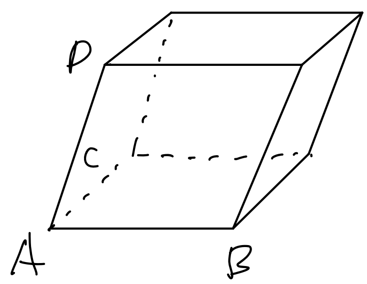
$=$ Volume of the parallelepiped.

<u>Remarks</u>: (i) $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \Leftrightarrow$ Vol(parallelepiped) $= 0$

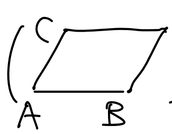
$\Leftrightarrow \{\vec{a}, \vec{b}, \vec{c}\}$ are linearly dependent.

(ii) Tetrahedron





Vol(Tetrahedron)

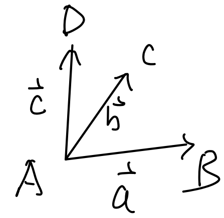
$= \frac{1}{3}$ Area($\triangle ABC$) $\cdot$ height $= \frac{1}{3} \cdot \frac{1}{2}$ Area() $\cdot$ height

$= \frac{1}{6}$ Vol(Parallelepiped)

$= \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|$

eg Let $A=(1,0,1)$, $B=(1,1,2)$, $C=(2,1,1)$, $D=(2,1,3)$

Find volume of tetrahedron $ABCD$



Soln: $\overrightarrow{AB} = (1,1,2)-(1,0,1)$

$= (0,1,1)$

$\overrightarrow{AC} = (2,1,1)-(1,0,1) = (1,1,0)$

$\overrightarrow{AD} = (2,1,3)-(1,0,1) = (1,1,2)$

$$\text{Vol(Tetrahedron)} = \frac{1}{6}\left|(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}\right|$$

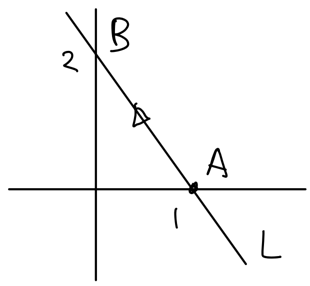
$$= \frac{1}{6}\left|\det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\right| = \frac{1}{6}\left|-2\right| = \frac{1}{3}$$

Linear Objects in $\mathbb{R}^n$

(lines, plane, k-plane, hyperplane)

Line

eg in $\mathbb{R}^2$



Equation form

$$2x+y=2$$

Parametric form

$$(x,y) = \overrightarrow{OA} + t\,\overrightarrow{AB}$$
$$= (1,0) + t[(0,2)-(1,0)]$$
$$= (1,0) + t(-1,2)$$
$$= (1-t,\ 2t)$$

ie $\begin{cases} x = 1-t \\ y = 2t \end{cases}$ $\quad t \in \mathbb{R}$

Note : Symmetric / Slope form $\quad \dfrac{x-1}{-1} = \dfrac{y-0}{2}$