Math 2010 E Advanced Calculus I

Differential Calculus of Functions of Several variables.

$$f(x,y), \quad f(x,y,z), \quad \underline{f(x_1,\cdots,x_n)}$$

$\underbrace{\qquad\qquad}$
mainly

$n = 2, 3$ (most of the times)

<u>Vectors in $\mathbb{R}^n$</u> : $\vec{X} = (x_1,\cdots,x_n) \in \mathbb{R}^n$

$$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} \quad (n \text{ copies of } \mathbb{R})$$

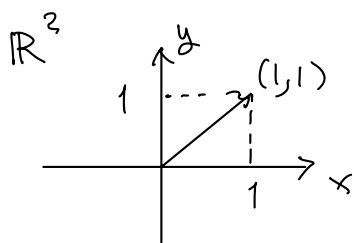$$= \{(x_1,\cdots,x_n): x_i \in \mathbb{R} \text{ for } i=1,\cdots,n\}$$

<u>Remarks</u> : (i) In Textbook, "bold face" $\mathbf{X}$ is used to denote a vector.

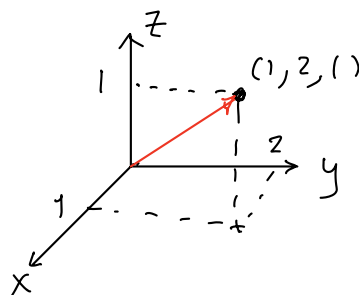(ii) In high school, $\overrightarrow{AB}$ is a vector with initial point A & terminal point B.

For $\vec{X} = (x_1,\cdots,x_n)$ is a vector with initial point $A = 0 = (0,\cdots,0)$ & terminal point $B = (x_1,\cdots,x_n)$.

(iii) $\mathbb{R}^n$ is called <u>Cartesian product</u> of n-copies of $\mathbb{R}$.

$(x_1,\cdots,x_n)$ Cartesian/rectangular coordinates of the point.

eg: $\mathbb{R}^2$

$\mathbb{R}^3$

Basic Operations of Vectors

Let $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$

$\vec{b} = (b_1, \cdots, b_n) \in \mathbb{R}^n$

$r \in \mathbb{R}$

Equality

$$\vec{a} = \vec{b} \iff a_i = b_i \quad \text{for } i = 1, \cdots, n.$$

Addition

$$\vec{a} + \vec{b} = (a_1 + b_1, \cdots, a_n + b_n)$$
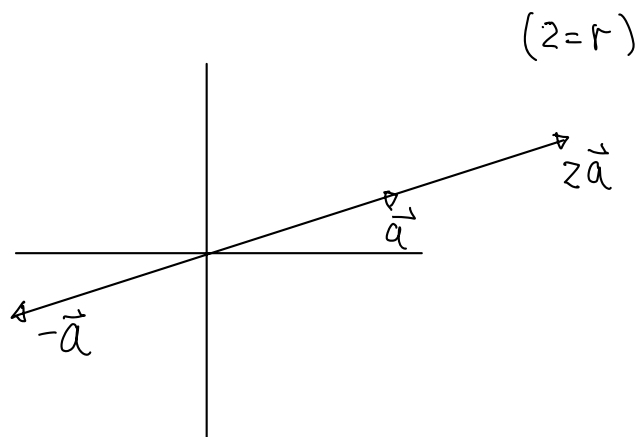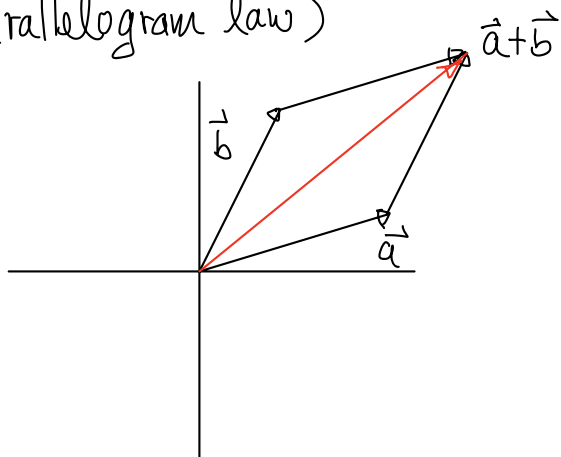
Scalar Multiplication

$$r\vec{a} = (ra_1, \cdots, ra_n)$$

Subtraction

$$\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$$
$$= (a_1 - b_1, \cdots, a_n - b_n)$$

(Parallelogram law)

(2 = r)

(Ex: what is $\vec{a} - \vec{b}$?)

Length, Dot (Inner) Product in $\mathbb{R}^n$

Defn :   Let $\vec{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$

$\qquad\qquad \vec{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$

Then   **dot product**   (or **inner product**)

$$\vec{a} \cdot \vec{b} = a_1 b_1 + \dots + a_n b_n$$

$\uparrow$

(Have to write the dot !)

In particular,

$$\vec{a} \cdot \vec{a} = a_1^2 + \dots + a_n^2 \overset{def}{=} \|\vec{a}\|^2$$

$\|\vec{a}\| = \sqrt{a_1^2 + \dots + a_n^2}$ is called the **length/magnitude** of

$\qquad\qquad$ the vector $\vec{a}$.

$\qquad$ (In Textbook, it's denoted by $|\vec{a}|$.)

Remark :   Another notation for dot (inner) product

$$\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b}$$

Properties of Dot Product

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$, $r \in \mathbb{R}$. Then

(1) $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

$\qquad\quad \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

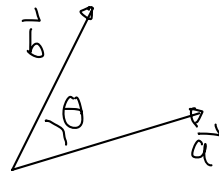(2) $(r\vec{a}) \cdot \vec{b} = \vec{a} \cdot (r\vec{b}) = r(\vec{a} \cdot \vec{b})$

(3)　$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(4)　$\vec{a} \cdot \vec{a} \geq 0$　and　"equality holds" $\Leftrightarrow$ $\vec{a} = \vec{0} = (0, \cdots, 0)$　← zero vector

(5)　$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

(6)　$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

where θ = angle between $\vec{a}$ & $\vec{b}$

Hence, if $\vec{a}, \vec{b} \neq \vec{0}$,

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \vec{a} \perp \vec{b} \text{ (perpendicular)}$$

<u>Remarks</u> (i) For $n = 2, 3$, (5) & (6) are geometric properties of $\mathbb{R}^n$.
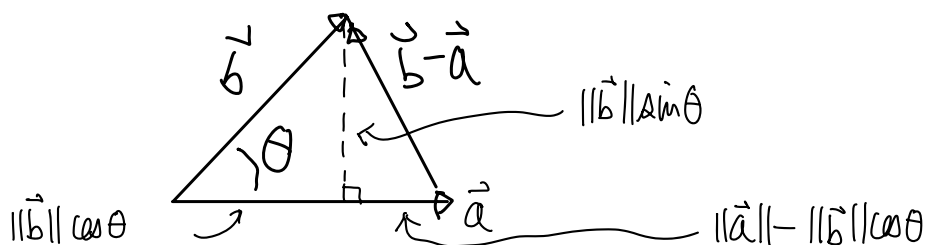
(ii) For $n \geq 4$, $\|\vec{a}\|$ in (5) & (6) is defined before and the θ in (6) is the definition

$$\theta \stackrel{def}{=} \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \right) \qquad \text{for high dimensional vectors}$$

(iii) If $\|\vec{u}\| = 1$, then $\vec{u}$ is called a <u>unit vector</u>.

eg: $\left. \begin{array}{l} \hat{i} = (1, 0, 0) \\ \hat{j} = (0, 1, 0) \\ \hat{k} = (0, 0, 1) \end{array} \right\}$ are unit vectors in $\mathbb{R}^3$.

<u>Pf of (6)</u> (for $n \leq 3$)　(Assume $\theta \leq \frac{\pi}{2}$)

Note $\|\vec{b}-\vec{a}\|^2 = (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a})$ (by (5))

$\qquad = \vec{b}\cdot(\vec{b}-\vec{a}) - \vec{a}\cdot(\vec{b}-\vec{a})$ (by (1) & (2))

$\qquad = (\vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{a}) - (\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{a})$

$\qquad = \|\vec{b}\|^2 - 2\,\vec{a}\cdot\vec{b} + \|\vec{a}\|^2$ (by (3) & (5))

From the figure

$\|\vec{b}-\vec{a}\|^2 = \left(\|\vec{a}\| - \|\vec{b}\|\cos\theta\right)^2 + \left(\|\vec{b}\|\sin\theta\right)^2$

$\qquad = \|\vec{a}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta + \|\vec{b}\|^2\cos^2\theta + \|\vec{b}\|^2\sin^2\theta$

$\qquad = \|\vec{a}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta + \|\vec{b}\|^2$

$\therefore \quad \|\vec{a}\|\,\|\vec{b}\|\cos\theta = \vec{a}\cdot\vec{b}\ .$ ※

eg: Suppose $\vec{v}$ & $\vec{w}$ are vectors of the <u>same length</u>.
Show that $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = 0$.
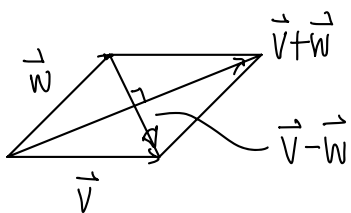
Soln: $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{v} - \vec{v}\cdot\vec{w} - \vec{w}\cdot\vec{w}$

$\qquad = \|\vec{v}\|^2 - \|\vec{w}\|^2$

$\qquad = 0 \qquad$ by assumption that $\vec{v}$ & $\vec{w}$ have same length

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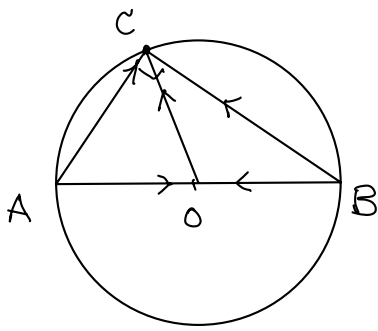
<u>Remark</u>: $(\vec{v}+\vec{w})\perp(\vec{v}-\vec{w})$ provided $\|\vec{v}\| = \|\vec{w}\|$,

Geometric meaning:

Diagonals of a rhombus are perpendicular.

eg:

AB is diameter, C on the circle
show that $\angle ACB = 90° = \frac{\pi}{2}$

Soln: $\vec{AC} = \vec{AO} + \vec{OC}$ (check)

$\vec{BC} = \vec{BO} + \vec{OC}$

$AB = $ diameter $\Rightarrow \vec{BO} = -\vec{AO}$

Then $\vec{AC} \cdot \vec{BC} = (\vec{AO} + \vec{OC}) \cdot (\vec{BO} + \vec{OC})$

$$= (\vec{AO} + \vec{OC}) \cdot (-\vec{AO} + \vec{OC})$$

$$= -\|\vec{AO}\|^2 - \vec{OC} \cdot \vec{AO} + \vec{AO} \cdot \vec{OC} + \|\vec{OC}\|^2$$

$$= -\|\vec{AO}\|^2 + \|\vec{OC}\|^2$$

$$= 0 \qquad \text{since } \|\vec{OC}\| = \|\vec{AO}\| = \text{radius.}$$

$$\therefore \quad \angle ACB = \frac{\pi}{2}$$

※

Two inequalities (Au's Book §1.2)

Cauchy-Schwarz Inequality

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$

i.e. $\left| \sum_{i=1}^{n} a_i b_i \right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}$

$$\boxed{\begin{array}{l} \text{Equality holds} \qquad\qquad \vec{a} = r\vec{b} \quad \text{or} \quad \vec{b} = r\vec{a} \\[4pt] |\vec{a}\cdot\vec{b}| = \|\vec{a}\|\|\vec{b}\| \qquad \Longleftrightarrow \qquad \text{for some } r \in \mathbb{R} \end{array}}$$
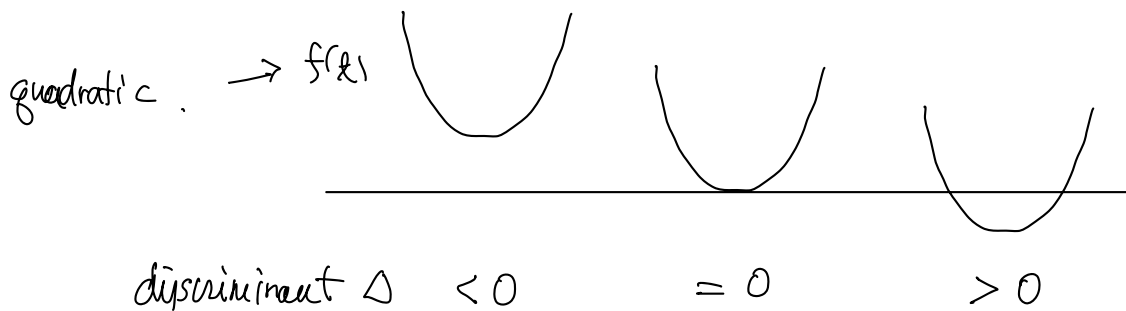
Pf: <u>Case 1</u> If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then both sides are 0

  equality holds $\vec{a} = 0\vec{b}$ or $\vec{b} = 0\vec{a}$.

<u>Case 2</u> $\vec{a} \neq \vec{0}$ & $\vec{b} \neq \vec{0}$.

  Let $f(t) = \|t\vec{a} - \vec{b}\|^2$ for $t \in \mathbb{R}$.

  Then $f(t) = (t\vec{a} - \vec{b})\cdot(t\vec{a} - \vec{b})$

$$= t^2\|\vec{a}\|^2 - 2t\vec{a}\cdot\vec{b} + \|\vec{b}\|^2 \quad (\geq 0)$$

  Note $f(t) = \|t\vec{a} - \vec{b}\|^2 \geq 0 \quad , \forall t \in \mathbb{R}$

quadratic. $\rightarrow f(t)$

discriminant $\triangle$ < 0 \qquad $= 0$ \qquad > 0

$\therefore \quad t^2\|\vec{a}\|^2 - 2t\vec{a}\cdot\vec{b} + \|\vec{b}\|^2 \geq 0$

$\Rightarrow \qquad \triangle \leq 0$

i.e. $(-2\vec{a}\cdot\vec{b})^2 - 4(\|\vec{a}\|^2)(\|\vec{b}\|^2) \leq 0$

$\Rightarrow \qquad |\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$

Equality holds $\iff$ $\triangle = 0$

$\iff$ $f(t)$ has a repeated root $r \in \mathbb{R}$

$\iff$ $\| r\vec{a} - \vec{b} \|^2 = 0$

$\iff$ $r\vec{a} = \vec{b}$.   ※