MATH1050A Assignment 3

1. Prove the statements below:

   (a) *Suppose* ζ, η *are complex numbers. Then* $\overline{\zeta\eta} = \bar{\zeta} \cdot \bar{\eta}$.

   (b) *Suppose* ζ *is a complex number. Then* $|\zeta|^2 = \zeta\bar{\zeta}$.

   (c) *Let* ζ, η *be complex numbers. Suppose* $\eta \neq 0$. *Then* $\dfrac{\zeta}{\eta} = \dfrac{\zeta\bar{\eta}}{|\eta|^2}$.

   (d) *Suppose* ζ, η *are complex numbers. Then* $|\zeta\eta| = |\zeta| \cdot |\eta|$.

2. Let $\omega = \dfrac{\sqrt{3}+i}{2}$.

   (a) Write down the respective values of ω^2, ω^3, ω^{11}, ω^{12}.

   (b) Hence, or otherwise, find the value of $\displaystyle\sum_{k=0}^{2230} \omega^{k+1}$.

3. Let a, b, c be real numbers. Suppose $a^2 + b^2 + c^2 = 1$ and $c \neq 1$. Define $z = \dfrac{a+bi}{1-c}$.

   (a) Express $|z|^2$ in terms of c alone.

   (b) Express each of a, b, c in terms of $z, \bar{z}$ alone.

4. (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (A), which is known as the **Parallelogramic Identity**. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (A) *Suppose* z, w *are complex numbers. Then* $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.

   > Suppose z, w are complex numbers.
   > Then $|z+w|^2 = (z+w)\overline{(z+w)} =$ ______(I)______.
   > Also, $|z-w|^2 = |z+(-w)|^2 =$ ______(II)______.
   > Then $|z+w|^2 + |z-w|^2 =$ ______(III)______.

   (b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (B). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (B) *Suppose* r, s, t *are complex numbers. Then* $|2r-s-t|^2 + |2s-t-r|^2 + |2t-r-s|^2 = 3(|s-t|^2 + |t-r|^2 + |r-s|^2)$.

   > Suppose ______(I)______.
   > Then, by the Parallelogramic Identity,
   > $$|2r-s-t|^2 = |(r-s)+(r-t)|^2 = \underline{\qquad\text{(II)}\qquad} = 2|r-s|^2 + 2|t-r|^2 - |s-t|^2.$$
   > Similarly, $|2s-t-r|^2 =$ ______(III)______.
   > Also, ______(IV)______.
   > Therefore $|2r-s-t|^2 + |2s-t-r|^2 + |2t-r-s|^2 =$ ______(V)______.

   (c)$^\diamond$ Applying the Parallelogramic Identity, or otherwise, prove the statement below.

   - *Let* ζ, α, β *be complex numbers. Suppose* $\zeta^2 = \alpha^2 + \beta^2$. *Then* $|\zeta+\alpha| + |\zeta-\alpha| = |\zeta+\beta| + |\zeta-\beta|$.

5. Let α be a non-zero complex number. Suppose $\alpha^5/\overline{\alpha}^3 = 4$. Also suppose that α is neither real nor purely imaginary.

   (a) What is the modulus of α?

   (b) Find all possible values of α. Express your answers in standard form.

6. *There is no need to give any justifications for your answers in this question.*

Find all the solutions of the system of equations $\begin{cases} |z-2-2i| &= 2 \\ |z-4+2i| &= |z-2i| \end{cases}$ with unknown z in $\mathbb{C}$.

Remark. What are the curves described by the respective equations in the Argand plane?

7. *There is no need to give any justifications for your answers in this question.*

Consider the system of equations $(S_{\alpha,r})$: $\begin{cases} |z-2i| &= 2 \\ |z-4-4i| &= |z| \\ |z-\alpha| &= r \end{cases}$ with unknown z in $\mathbb{C}$. Here α is some complex number and r is a non-negative real number.

Suppose that $(S_{\alpha,r})$ has two distinct solutions.

(a) Write down all solutions of $(S_{\alpha,r})$.

(b) What is the smallest possible value of r?

(c) What is the value of α if $|\mathsf{Re}(\alpha)| = |\mathsf{Im}(\alpha)|$?

Remark. What are the curves described by the respective equations in the Argand plane?

8. Let $z = \dfrac{2(\sqrt{3}-i)}{\sqrt{3}+i}$.

(a) Express z in polar form.

(b) Hence, or otherwise, find the numbers below. Express the respective answers in standard form.

i. z^{2020}.

ii. The square roots of z.

9. Let $f(z)$ be the quadratic polynomial given by $f(z) = z^2 + (-4-6i)z + [-3 + (12-2\sqrt{3})i]$.

(a) Apply the method of completing the square, express the polynomial $f(z)$ in the form $f(z) = \kappa(z+\lambda)^2 + \mu$ for some appropriate complex numbers κ, λ, μ.

(b) Hence determine all the roots of $f(z)$ in $\mathbb{C}$.

10. *In this question, we take for granted everything that we learnt about* polynomials *and* polynomial equations *in school mathematics.*

We introduce the definitions for the notions of *algebraicity* and *transcendence* for complex numbers:

- *Let α be a complex number. We say that α is* **algebraic** *if there exists some non-constant polynomial $f(x)$ whose coefficients are rational numbers such that $f(\alpha) = 0$.*
- *Let τ be a complex number. We say that τ is* **transcendental** *if τ is not algebraic.*

(a) Verify that the numbers below are algebraic:

i. 0.

ii. 1.

iii. i.

iv. $\sqrt{2}$.

v. $\sqrt{2}i$.

vi.$^\diamondsuit$ $\sqrt{2}+\sqrt{3}$.

vii.$^\diamondsuit$ $\sqrt{2}+i$.

viii.$^\diamondsuit$ $\sqrt{5+\sqrt[3]{2}}$.

Remark. According to the definition for the notion of algebraicity, in each part all you need to do in the argument is to name an appropriate non-constant polynomial whose coefficients are rational numbers for which the number concerned is a root of the polynomial. How you conceive that polynomial is not part of the argument. That said, for the more difficult part, it may be good to do some appropriate roughwork on the search for an appropriate polynomial to be named.

Further remark. In general, it is difficult to prove a given number is transcendental by elementary methods.

(b) Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (C) and a proof for the statement (D). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

i. Here we prove Statement (C):

(C) *Let α be a positive real number. Suppose α is algebraic. Then $\sqrt{\alpha}$ is algebraic.*

Let α be ______(I)______ . ___(II)___ α is algebraic.

By the definition for the notion of algebraicity, ______(III)______ such that $f(\alpha) = 0$.

Denote the degree of $f(x)$ by k. Since $f(x)$ is non-constant, we have $k \geq 1$.

For the same $f(x)$, ______(IV)______ such that a_k is ___(V)___ and $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ as polynomials.

Define the polynomial $g(x)$ by ______(VI)______ .

By definition, $g(x)$ is a polynomial whose coefficients are ___(VII)___ .

Since a_k is ___(VIII)___ , the degree of $g(x)$ is $2k$. Then ___(IX)___ is non-constant.

We have $g(\sqrt{\alpha}) =$ ______(X)______ .

Therefore, by definition, ______(XI)______ .

ii. Here we apply the method of proof-by-contradiction to prove Statement (D) (with the help of Statement (C)):

(D) *Let τ be a positive real number. Suppose τ is transcendental. Then τ^2 is transcendental.*

Let τ be a positive real number. ______(I)______

Further suppose ______(II)______ .

Then by definition, since τ^2 was not transcendental, ______(III)______ .

Note that τ^2 is a positive real number. Then, by the result in Statement (C), ______(IV)______ .

Note that $\sqrt{\tau^2} =$ ___(V)___ . Then τ would be algebraic.

Now τ is simultaneously ______(VI)______ .

Contradiction arises.

It follows, in the first place, that ______(VII)______ .

11. *This is a review question on* geometric progressions.

(a) Fill in the blanks in the passage below so as to give the definition for the notion of *geometric progression*:

Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence of non-zero complex numbers.

The infinite sequence $\{b_n\}_{n=0}^{\infty}$ is said to be a **geometric progression** *if the statement (GP) holds:*

(GP) ______(I)______ such that ______(II)______ .

The number r is called the **common ratio** *of this geometric progression.*

(b) Consider the statement (E):

(E) *Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence of non-zero complex numbers. Suppose $\{b_n\}_{n=0}^{\infty}$ is a geometric progression. Then there exists some non-zero complex number r such that for any $m \in \mathbb{N}$, $b_m = b_0 r^m$.*

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (E). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

Let $\{b_n\}_{n=0}^{\infty}$ is an infinite sequence of non-zero complex numbers. Suppose $\{b_n\}_{n=0}^{\infty}$ is a geometric progression.

Then ______(I)______ .

Pick any $m \in \mathbb{N}$.

We have ___(II)___ , $\dfrac{b_2}{b_1} = r$, $\dfrac{b_3}{b_2} = r$, ..., $\dfrac{b_{m-1}}{b_{m-2}} = r$, and ___(III)___ .

Then $\dfrac{b_m}{b_0} =$ ______(IV)______ .

Therefore ___(V)___ .

(c) Hence, or otherwise, prove the statement $(\sharp)$:

$(\sharp)$ *Let $\{a_n\}_{n=0}^{\infty}$ be a geometric progression. Suppose $k, \ell, m \in \mathbb{N}$, and $a_k = A$, $a_\ell = B$ and $a_m = C$. Then $A^{\ell-m} B^{m-k} C^{k-\ell} = 1$.*

12. *This is a review question on* arithmetic progressions.

(a) Consider the statement (F):

(F) *Let a, b, c be complex numbers. Suppose a, b, c are in arithmetic progression. Then $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression.*

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (F). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> Let a, b, c be complex numbers. ______(I)______
>
> Denote the common difference by ___(II)___ .
>
> [We want to verify that $a^2 - bc, b^2 - ca, c^2 - ab$.
> By definition, it sufficies to verify that $(b^2 - ca) - (a^2 - bc) = (c^2 - ab) - (b^2 - ca)$.]
>
> By definition, we have $b - a = d$ and ______(III)______ .
>
> Note that $(b^2 - ca) - (a^2 - bc) =$ ______(IV)______.
>
> Also note that ______(V)______ .
>
> Then ______(VI)______ .
>
> Hence $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression.

(b) Prove the statement ($\sharp$):

($\sharp$) *Let a, b, c be complex numbers. Suppose $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression. Further suppose $a + b + c \neq 0$. Then a, b, c are in arithmetic progression.*

Remark. You may take for granted the result ($\natural$):

($\natural$) *Suppose u, v, w are numbers. Then u, v, w are in arithmetic progression iff $v = \dfrac{u+w}{2}$.*

13. Let s, t, u, v be non-zero complex numbers. Suppose $\dfrac{1}{s}, \dfrac{1}{t}, \dfrac{1}{u}, \dfrac{1}{v}$ form an arithmetic progression.

    (a) Prove that $t = \dfrac{2su}{s+u}$.

    (b) Prove that $\dfrac{3s-u}{s+u} = \dfrac{t}{v}$.

14. Let r be a number, not equal to 1. For each positive integer n, define $s_n = 1 + r + r^2 + \cdots + r^{n-1}$.

    Prove that $\dfrac{s_1 + s_2 + s_3 + \cdots + s_{n-1} + s_n}{n+1} = \dfrac{1 - s_{n+1}/(n+1)}{1-r}$ for any positive integer n.

15. Let n be a positive integer.

    (a) Let x, t be real numbers. Suppose $t \neq 0$.

    Express $\dfrac{(x+t)^n - x^n}{t}$ as a sum of n terms, each of the form $(x+t)^j x^k$ for some appropriate integer j, k.

    (b) Hence, or otherwise, deduce the statements below:

    i. $\displaystyle\lim_{t \longrightarrow 0} \frac{(x+t)^n - x^n}{t} = nx^{n-1}$ *for any $x \in \mathbb{R}$.*

    ii. $\displaystyle\lim_{t \longrightarrow 0} \frac{1}{t}\left[\frac{1}{(x+t)^n} - \frac{1}{x^n}\right] = -\frac{n}{x^{n+1}}$ *for any $x \in (0, +\infty)$.*

    Remark. Take for granted the result ($\sharp$):

    ($\sharp$) *For each positive integer k, for any $u \in \mathbb{R}$, $\displaystyle\lim_{s \longrightarrow 0} (u+s)^k = u^k$.*

16. Let x, θ be real numbers. Suppose $|\sin(\theta)| < 1$.

    (a) Show that

    $$\begin{aligned} & 1 + (1+x)\sin(\theta) + (1+x+x^2)\sin^2(\theta) + \cdots + (1+x+x^2+\cdots+x^n)\sin^n(\theta) \\ = & \frac{A}{B-x} \cdot \frac{C - \sin^{n+H}(\theta)}{D - \sin(\theta)} - \frac{x}{E-x} \cdot \frac{F - x^{n+I}\sin^{n+J}(\theta)}{G - x\sin(\theta)} \end{aligned}$$

    for each positive integer n.

    Here $A, B, C, D, E, F, G, H, I, J$ are some appropriate integers, whose values you have to determine explicitly.

    (b) Suppose $|x\sin(\theta)| < 1$. Evaluate the limit

    $$\lim_{n \longrightarrow \infty} [1 + (1+x)\sin(\theta) + (1+x+x^2)\sin^2(\theta) + \cdots + (1+x+x^2+\cdots+x^n)\sin^n(\theta)].$$