Can we express the matrix inverse of an invertible matrix with determinants?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

For the moment, we do not suppose A to be invertible.

$$\begin{cases} a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) + a_{13}\det(A(1|3)) - a_{14}\det(A(1|4)) = \det(A) \\ a_{21}\det(A(1|1)) - a_{22}\det(A(1|2)) + a_{23}\det(A(1|3)) - a_{24}\det(A(1|4)) = \det\left(\begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\right) = 0 \\ a_{31}\det(A(1|1)) - a_{32}\det(A(1|2)) + a_{33}\det(A(1|3)) - a_{34}\det(A(1|4)) = \det\left(\begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\right) = 0 \\ a_{41}\det(A(1|1)) - a_{42}\det(A(1|2)) + a_{43}\det(A(1|3)) - a_{44}\det(A(1|4)) = \det\left(\begin{bmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\right) = 0 \end{cases}$$

We now introduce some special notations to re-present the above equalities.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Write $\alpha_{ij} = (-1)^{i+j} \det(A(i|j))$ for each i, j.

Done already:

$$\begin{cases} a_{11}\alpha_{11} + a_{12}\alpha_{12} + a_{13}\alpha_{13} + a_{14}\alpha_{14} = \det(A) \\ a_{21}\alpha_{11} + a_{22}\alpha_{12} + a_{23}\alpha_{13} + a_{24}\alpha_{14} = 0 \\ a_{31}\alpha_{11} + a_{32}\alpha_{12} + a_{33}\alpha_{13} + a_{34}\alpha_{14} = 0 \\ a_{41}\alpha_{11} + a_{42}\alpha_{12} + a_{43}\alpha_{13} + a_{44}\alpha_{14} = 0 \end{cases}$$

Imitating what has been done, we obtain:

$$\begin{cases} a_{11}\alpha_{21} + a_{12}\alpha_{22} + a_{13}\alpha_{23} + a_{14}\alpha_{24} = 0 \\ a_{21}\alpha_{21} + a_{22}\alpha_{22} + a_{23}\alpha_{23} + a_{24}\alpha_{24} = \det(A) \\ a_{31}\alpha_{21} + a_{32}\alpha_{22} + a_{33}\alpha_{23} + a_{31}\alpha_{24} = 0 \\ a_{41}\alpha_{21} + a_{42}\alpha_{22} + a_{43}\alpha_{23} + a_{44}\alpha_{24} = 0 \end{cases}$$

Further imitating what has been done, we obtain:

$$\begin{cases} a_{11}\alpha_{31} + a_{12}\alpha_{32} + a_{13}\alpha_{33} + a_{14}\alpha_{34} = 0 \\ a_{21}\alpha_{31} + a_{22}\alpha_{32} + a_{23}\alpha_{33} + a_{24}\alpha_{34} = 0 \\ a_{31}\alpha_{31} + a_{32}\alpha_{32} + a_{33}\alpha_{33} + a_{34}\alpha_{34} = \det(A) \\ a_{41}\alpha_{31} + a_{42}\alpha_{32} + a_{43}\alpha_{33} + a_{44}\alpha_{34} = 0 \end{cases}$$

And also:

$$\begin{cases} a_{11}\alpha_{41} + a_{12}\alpha_{42} + a_{13}\alpha_{43} + a_{14}\alpha_{44} = 0 \\ a_{21}\alpha_{41} + a_{22}\alpha_{42} + a_{23}\alpha_{43} + a_{24}\alpha_{44} = 0 \\ a_{31}\alpha_{41} + a_{32}\alpha_{42} + a_{33}\alpha_{43} + a_{34}\alpha_{44} = 0 \\ a_{41}\alpha_{41} + a_{42}\alpha_{42} + a_{43}\alpha_{43} + a_{44}\alpha_{44} = \det(A) \end{cases}$$

We now introduce some further notations to 'condense' the above 4^2 equalities into one equality.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$ Write $\alpha_{ij} = (-1)^{i+j} \det(A(i|j))$ for each i, j.

Write $$\mathrm{Ad}(A) = \left[\begin{array}{c|c|c|c} \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \alpha_{42} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{43} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & \alpha_{44} \end{array}\right].$$ (This is called the adjoint of A.)

$$A \cdot \mathrm{Ad}(A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \left[\begin{array}{c|c|c|c} \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \alpha_{42} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{43} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & \alpha_{44} \end{array}\right]$$

$$= \begin{bmatrix} \det(A) & 0 & 0 & 0 \\ 0 & \det(A) & 0 & 0 \\ 0 & 0 & \det(A) & 0 \\ 0 & 0 & 0 & \det(A) \end{bmatrix} = \det(A) \cdot I_4.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$ Write $\alpha_{ij} = (-1)^{i+j} \det(A(i|j))$ for each i, j.

Write $$\mathrm{Ad}(A) = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \alpha_{42} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{43} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & \alpha_{44} \end{bmatrix}.$$

Now known: $A \cdot \mathrm{Ad}(A) = \det(A) \cdot I_4$.

Suppose $\det(A) \neq 0$. Then A is invertible, and

$$A \cdot \left(\frac{1}{\det(A)} \mathrm{Ad}(A) \right) = I_4$$

So the matrix inverse of A is given by

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Ad}(A).$$ (This is 'Cramer's formula'.)