Week 6:

Recall: Let A be a $n\times n$ matrix, the following are equivalent:

1. A is non-singular
2. A is invertible
3. A is row equivalent to I_n
4. $\exists B$ s.t. $BA = I_n$
5. $\exists H$ s.t. $HA = I_n$.

More about invertible:

compare with $\mathbb{R}$: $ab \neq 0 \iff a \neq 0$ and $b \neq 0$

invertible in $\mathbb{R}$

Thm: Let A, B be $n\times n$ matrix, then

AB = invertible iff A, B are both invertible.

pf: $(\Leftarrow)$: done before with

$$(AB)^{-1} = B^{-1}A^{-1}$$

$(\Rightarrow)$: If AB = invertible.

then $\exists\ (AB)^{-1} = C$ s.t. $CAB = ABC = I_n$

$(CA)B = I_n \Rightarrow B$ is invertible

$\Rightarrow B^{-1}$ exists

Hence $A = \underbrace{(AB)}_{\text{invertible}} \underbrace{B^{-1}}$ is invertible #.

Corollary: If $A_1, A_2, \cdots, A_N$ are $n \times n$ matrices s.t.

$A_1 \cdots A_N$ = invertible, then each A_i is invertible.

Corollary to thm of equivalence

The inverse of any invertible matrix is a product of finitely many row operation matrix

pf: If A is invertible, A is row eq. to I_n and hence

$\exists$ row op. $\{\rho_i\}_{i=1}^{N}$ s.t. $A \xrightarrow{\rho_1} A_1 \xrightarrow{\rho_2} A_2 \rightarrow \cdots \xrightarrow{\rho_N} I_n = A_N$

i.e. $\exists$ row op. matrix H_i corresponds to ρ_i

s.t. $H_N H_{N-1} \cdots H_1 \cdot A = I_n$

$\therefore A^{-1} = H_N H_{N-1} \cdots H_1$ #.

Example to find A^{-1}

① $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

$[A \mid I_2] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$

$\xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right]$

$= [I_2 \mid A^{-1}]$ #

③ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$$[A \mid I_3] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow[R_3 + R_1]{-2R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] = [I_3 \mid A^{-1}] \quad \#$$

Relation to sol of $LS(A,b)$:

Lemma 1: (Recall) Let A be $n \times n$ matrix.

If for all b, $n \times 1$ matrix, there is at most one sol. to $LS(A,b)$, then A is invertible.

Lemma 2: Let A be $n \times n$ matrix.

If for all b ($n \times 1$ matrix) there is at least one sol. to $LS(A, b)$, then A is invertible.

pf: take $b_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ i-th row.

$\exists\, x_i$ s.t. $Ax_i = b_i$, then

$$A \cdot [x_1 \ x_2 \ \cdots \ x_n] = [Ax_1 \ Ax_2 \ \cdots \ Ax_n]$$
$$= [b_1 \ b_2 \ \cdots \ b_n] = I_n$$

$\therefore$ take $G = [x_1 \cdots x_n]$ be $n \times n$ matrix s.t.

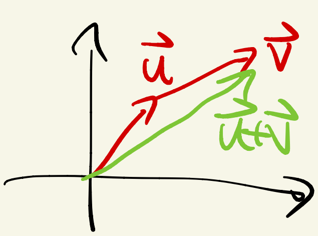
$$AG = I_n \Rightarrow A = \text{invertible}. \#$$

In $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$

we write $\vec{x}$ as $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, a $n \times 1$ matrix.

call it a column vector

why vector:



$=$ "displacement" (distance moved w/ direction)

Clearly, $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$.

We may consider
- $\vec{u}+\vec{v}$
- $\alpha\vec{u}$

still a column vector

The following is true:

① $\vec{u}+\vec{v} \in \underline{\mathbb{R}^n}$ ③ $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

② $\alpha\vec{u} \in \underline{\mathbb{R}^n}$ ④ $\vec{u}+0 = \vec{u}$

⑤ $\vec{u}+(-\vec{u}) = \vec{0}$ ⑥ $\alpha(\beta\vec{u}) = \beta(\alpha\vec{u}) = (\alpha\beta)\vec{u}$

⑦ $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$ ⑧ $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

⑨ $(\alpha+\beta)\vec{u} = \alpha\vec{u}+\beta\vec{u}$ ⑩ $1\cdot\vec{u} = \vec{u}$.

they follow from def. of matrix.

Concept of vector space:

Defn: A set V with operation $+$, and scalar multiplication is a vector space if ①–⑩ holds.

Eg: ① The set of column vector $\mathbb{R}^n$

② The set of row vector ($1\times n$ matrix)

③ The set of matrix.

④ The set of polynomial of degree $\leq n$. P_n

- $(f+g)(x) \triangleq \sum_{i=0}^{n} (a_i+b_i) x^i$ where

$$f(x) = \sum_{i=0}^{n} a_i x^i \quad , g(x) = \sum_{i=0}^{n} b_i x^i .$$

- $(\alpha f)(x) \triangleq \sum_{i=0}^{n} \alpha a_i x^i$

- Inverse $(-f)(x) \triangleq \sum_{i=0}^{n} (-a_i) x^i$. , etc...

Special case: Given a vector space V,

want to study subset W of V s.t. W is also a vector.

Defn: A subset $W \subseteq V$ is a subspace if

① $W \neq \phi$

② $\forall w, v \in W$, $v+w \in W$

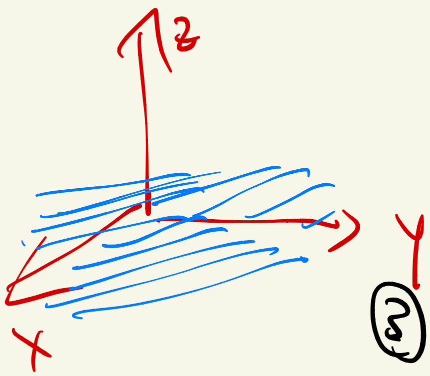
③ $\forall \alpha \in \mathbb{R}$, $\alpha v \in W$.

Eg: (i) $V = \mathbb{R}^n$, $W = \{0\}$ (trivial example)

(ii) $V = \mathbb{R}^n$, $W = \mathbb{R}^n$ (trivial example)

(iii) $V = \mathbb{R}^3$, $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x = 0 \right\}$ ~~x_3~~ (plane)

Since ① $W \neq \phi$, ② If $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\tilde{w} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} \in W$



then $w + \tilde{w} = \begin{bmatrix} x+\tilde{x} \\ y+\tilde{y} \\ z+\tilde{z} \end{bmatrix}$ with $x + \tilde{x} = 0$.

③ $\forall \alpha \in \mathbb{R}$, $\alpha v = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \end{bmatrix}$ with $\alpha x = \alpha \cdot 0 = 0$.
If $v \in W$.

(iv) $V = \mathbb{R}^3$, $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x+y+z=0 \right\}$

(skew planes)

(v) ☆☆ Null space of a matrix A, $N(A)$

If A = $m \times n$ matrix, then $N(A)$ is a subspace of $\mathbb{R}^n$.

<u>Because</u>: ① $0 \in N(A)$

② If $x, y \in N(A)$,

$$Ax + Ay = A(x+y) = 0$$

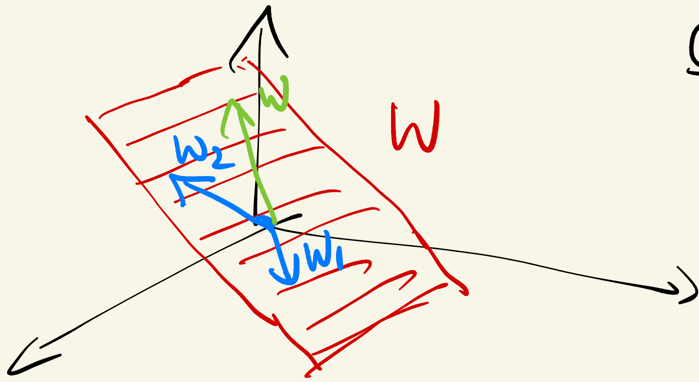
$$\Rightarrow x+y \in N(A).$$

③ If $\alpha \in \mathbb{R}$, $x \in N(A)$,

then $\alpha A x = A(\alpha x) = 0$

$$\Rightarrow \alpha x \in N(A). \#$$

Example : $V = \mathbb{R}^3$, $W =$ skew plane.



Q: how to represent W using vector ?? (instead of algebraic equation)

Choose $w_1, w_2 \in W$ s.t. $w_1 \nparallel w_2$.

then any $w_3 \in W$ is in form of

$$w_3 = \underbrace{\alpha w_1 + \beta w_2}_{} \text{ for some } \alpha, \beta \in \mathbb{R}.$$

linear combination of $w_1, w_2 \in W \leq \mathbb{R}^3$.

$$\therefore W = \{\alpha w_1 + \beta w_2 \mid \alpha, \beta \in \mathbb{R}\}$$

familiar ??

Number of parameter = 2.

And $W =$ subspace of $V = \mathbb{R}^3$.

Defn : Given $w_1, w_2, \cdots, w_m \in \mathbb{R}^n$, $\alpha_1, \alpha_2, \cdots, \alpha_m \in \mathbb{R}$

the linear combination is the vector in form

of $w = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m \in \mathbb{R}^n$.

And $W = \{\alpha_1 w_1 + \cdots + \alpha_m w_m \mid \alpha_1, \cdots, \alpha_m \in \mathbb{R}\}$

will be a sub-space of $\mathbb{R}^n$. (Keep the question in mind: what mean by $w_i \nparallel w_j$)

since ① $0 \in W$ (taking $\alpha_i = 0$) here ??

② if $\begin{cases} w = \sum_{i=1}^{m} \alpha_i w_i \\ \tilde{w} = \sum_{i=1}^{m} \tilde{\alpha}_i w_i \end{cases}$ then $w + \tilde{w} = \sum_{i=1}^{m} (\alpha_i + \tilde{\alpha}_i) w_i \in W$.

③ $\beta w = \sum_{i=1}^{m} (\beta \alpha_i) w_i \in W$.

Usually, denote W by $\text{span}\{w_1, \dots, w_m\}$ or $\langle w_1, \dots, w_m \rangle$.

Eg: $w_1 = \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}$, $w_3 = \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix}$

Question: Is $\begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$ inside $\langle w_1, w_2, w_3 \rangle$??

Ans: Yes. consider the Equ:

$$* \quad \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix} = b$$

Try to solve for $\alpha_1, \alpha_2, \alpha_3$.

$$* \iff \begin{cases} -7\alpha_1 - 6\alpha_2 - 12\alpha_3 = -33 \\ 5\alpha_1 + 5\alpha_2 + 7\alpha_3 = 24 \\ \alpha_1 \qquad\quad + 4\alpha_3 = 5. \end{cases} \quad (\iff LS(A, b))$$

The $LS(A,b)$ is solvable with $\begin{cases} \alpha_1 = -3 \\ \alpha_2 = 5 \\ \alpha_3 = 2 \end{cases}$

Q: Which column vector b is inside the span??

$\Updownarrow$

Which b we can solve $LS(A,b)$??

Thm Let A be a $m \times n$ matrix.

write $A = [A_1 | \cdots | A_n]$ s.t. A_i is the i-th column of A. (so that $A_i = m \times 1$ matrix)

Then $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a sol. to $Ax = b$ iff

$$b = x_1 A_1 + \cdots + x_n A_n.$$

pf: Straight forward by expansion.

(non)-Example: $W_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $W_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $W_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $W_4 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Q: Is $u = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ inside $\operatorname{span}\{W_1, W_2, W_3\}$??

Equivalent to solve $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 0 & 3 & 3 \end{array}\right]$

$$\xrightarrow{\text{Row op.}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = RREF$$

$\therefore$ No solution $\Rightarrow$ Not in the span!!

Q: Is u inside span$\{w_1, \cdots, w_4\}$??

Note: > 3

$\Leftrightarrow$ Solve $\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 0 & 3 & 1 & 3 \end{array}\right]$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 5/6 \\ 0 & 1 & -1 & 0 & -2/3 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right] = \text{RREF.}$$

$$\therefore \text{ solution set } = \left\{ \begin{bmatrix} 5/6 - t \\ -2/3 + t \\ t \\ 1/2 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 5/6 \\ -2/3 \\ 0 \\ 1/2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$\therefore$ take one sol. $\Rightarrow$

$$u = \frac{5}{6} w_1 - \frac{2}{3} w_2 + \frac{1}{2} w_4.$$

$\Rightarrow$ many options.

OR $= -\frac{1}{6} w_1 + \frac{1}{3} w_2 + w_3 + \frac{1}{2} w_4 = \cdots$

Q (Recall):

Given $v_1, v_2, \cdots, v_m \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$

(as a column vector), is $w \in \text{span}\{v_1, \cdots, v_m\}$??

$\Updownarrow$

Is $LS(A, w)$ solvable where

$A = [v_1 \mid v_2 \mid \cdots \mid v_m]$ (as a $n \times m$ matrix) ??

since $\Leftrightarrow$ finding $\alpha_1, \cdots, \alpha_m \in \mathbb{R}$ s.t.

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = w$$

$$\Leftrightarrow \left[v_1 \mid v_2 \mid \cdots \mid v_m \right] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = [w]$$

$n \times 1$ matrix

Motivation for the following.

Defn: Let A be $m \times n$ matrix

The column space of $A = \{ y \in \mathbb{R}^m \mid \text{there is } x \in \mathbb{R}^n \text{ s.t. } Ax = y \}$

Denote it by $C(A)$.

Rmk: from example above, there can be ∞ many x for a given y.

Thm: $C(A)$ is a subspace of $\mathbb{R}^m$.

pf: ① $0 \in C(A)$

② if $y_1, y_2 \in C(A)$,

then $\exists\, x_1, x_2 \in \mathbb{R}^n$ s.t.

$$\begin{cases} Ax_1 = y_1 \\ Ax_2 = y_2 \end{cases}$$

$\Rightarrow A(x_1 + x_2) = y_1 + y_2 \quad \therefore y_1 + y_2 \in C(A)$

③ if $\alpha \in \mathbb{R}$, $y \in C(A)$, then $\exists\, x$ s.t. $Ax = y$

then $A(\alpha x) = \alpha Ax = \alpha y$.

$\therefore \alpha y \in C(A)$. #

Illustration: Given $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 3 & 1 \end{bmatrix}$

then $C(A) = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_4 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : \alpha_i \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ #